1. Introduction

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ON THE CLASSIFICATION OF CERTAIN PIECEWISE LINEAR AND DIFFERENTIABLE MANIFOLDS IN DIMENSION EIGHT AND AUTOMORPHISMS OF $\#_{i=1}^{b}(S^2 \times S^5)$

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ABSTRACT. In this paper, we will be concerned with the explicit classification of closed, oriented, simply-connected spin manifolds in dimension eight with vanishing cohomology in the odd dimensions. The study of such manifolds was begun by Stefan Müller. In order to understand the structure of these manifolds, we will analyze their minimal handle presentations and describe explicitly to what extent these handle presentations are determined by the cohomology ring and the characteristic classes. It turns out that the cohomology ring and the characteristic classes do not suffice to reconstruct a manifold of the above type completely. In fact, the group $\mathrm{Aut}_0\big(\#_{i=1}^{b}(S^2 \times S^5)\big) / \mathrm{Aut}_0\big(\#_{i=1}^{b}(S^2 \times D^6)\big)$ of automorphisms of $\#_{i=1}^{b}(S^2 \times S^5)$ which induce the identity on cohomology modulo those which extend to $\#_{i=1}^{b}(S^2 \times D^6)$ acts on the set of oriented homeomorphy classes of manifolds with fixed cohomology ring and characteristic classes, and we will be also concerned with describing this group and some facts about the above action.

1. INTRODUCTION

The classification of topological manifolds up to homeomorphy is an extremely interesting and important problem. Let us restrict our attention to the case of closed (i.e., compact without boundary) and oriented simply connected manifolds. As a general classification scheme, surgery theory [1] solves this problem for manifolds within a given homotopy type, e.g., that of a sphere. Another approach to the classification "up to finite indeterminacy", using rational homotopy theory, is due to Sullivan [34]. Nevertheless, there are only a few explicit results which characterize the oriented homeomorphy type of a manifold in terms of easily computable invariants. They usually require many simplifying assumptions, such as high connectedness [36]. In this paper, we will consider *even cohomology manifolds* (or *E-manifolds*, for short) in dimension eight, by which we mean closed, oriented, simply connected, piecewise linear or smooth manifolds all of whose odd-dimensional homology groups with integer coefficients vanish. The universal coefficient theorem implies that all homology groups of an E-manifold are without torsion. Moreover, since $H^3(X, \mathbf{Z}_2) = 0$ for an E-manifold, two E-manifolds of dimension at least 6 are homeomorphic (as topological manifolds) if and only if they are piecewise linearly homeomorphic [16].

Though the class of E-manifolds is fairly restricted, it still contains many interesting examples from various areas of mathematics, such as the piecewise linear manifolds underlying the toric manifolds from Algebraic Geometry [4] or the polygon spaces [12], to mention a few. So far, E-manifolds have been classified up to dimension 6. Of course, in dimension 2 there is only S^2, in dimension 4, there is the famous classification result of Freedman various interesting aspects of which are discussed in [15], and finally in dimension 6, the classification was achieved by Wall [37] and Jupp [14]. Various applications of the latter result to Algebraic Geometry are surveyed in [26]. Finally, we refer to [2], [3], and [29] for the determination of projective algebraic structures on certain 6- and 8-dimensional E-manifolds.

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2. STATEMENT OF THE RESULT

We now discuss the main result of this note, namely the classification of E-manifolds of dimension 8 with vanishing second Stiefel-Whitney class in the form of an exact sequence of pointed sets. This result was motivated by the work [24]. In order to state it in a more elegant form, we will work with based manifolds. By a *based piecewise linear (smooth) E-manifold*, we mean a triple $(X, \underline{x}, \underline{y})$, consisting of a piecewise linear (smooth) E-manifold X and bases $\underline{x} = (x_1, \dots, x_{b_2(M)})$ for $H^2(X, \mathbf{Z})$ and $\underline{y} = (y_1, \dots, y_{b_4(M)})$ for $H^4(X, \mathbf{Z})$. Recall that by definition E-manifolds are oriented, so that the above data specify a basis for $H^*(X, \mathbf{Z})$, such that the bases for $H^i(X, \mathbf{Z})$ and $H^{8-i}(X, \mathbf{Z})$ are dual to each other with respect to the cup product. An *isomorphism between piecewise linear (smooth) based E-manifolds* $(X, \underline{x}, \underline{y})$ and $(X', \underline{x}', \underline{y}')$ is an orientation preserving piecewise linear (smooth) isomorphism $f\colon X \longrightarrow X'$ with $f^*(\underline{x}') = \underline{x}$