2. STATEMENT OF THE RESULT

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

consider *even cohomology manifolds* (or *E-manifolds*, for short) in dimension eight, by which we mean closed, oriented, simply connected, piecewise linear or smooth manifolds all of whose odd-dimensional homology groups with integer coefficients vanish. The universal coefficient theorem implies that all homology groups of an E-manifold are without torsion. Moreover, since $H^3(X, \mathbf{Z}_2) = 0$ for an E-manifold, two E-manifolds of dimension at least 6 are homeomorphic (as topological manifolds) if and only if they are piecewise linearly homeomorphic [16].

Though the class of E-manifolds is fairly restricted, it still contains many interesting examples from various areas of mathematics, such as the piecewise linear manifolds underlying the toric manifolds from Algebraic Geometry [4] or the polygon spaces [12], to mention a few. So far, E-manifolds have been classified up to dimension 6. Of course, in dimension 2 there is only S^2, in dimension 4, there is the famous classification result of Freedman various interesting aspects of which are discussed in [15], and finally in dimension 6, the classification was achieved by Wall [37] and Jupp [14]. Various applications of the latter result to Algebraic Geometry are surveyed in [26]. Finally, we refer to [2], [3], and [29] for the determination of projective algebraic structures on certain 6- and 8-dimensional E-manifolds.

ACKNOWLEDGEMENTS. My thanks go to J.-C. Hausmann for his interest in this work and for pointing out to me that an E-manifold of dimension eight is actually not determined by its classical invariants.

2. STATEMENT OF THE RESULT

We now discuss the main result of this note, namely the classification of E-manifolds of dimension 8 with vanishing second Stiefel-Whitney class in the form of an exact sequence of pointed sets. This result was motivated by the work [24]. In order to state it in a more elegant form, we will work with based manifolds. By a *based piecewise linear (smooth) E-manifold*, we mean a triple $(X, \underline{x}, \underline{y})$, consisting of a piecewise linear (smooth) E-manifold X and bases $\underline{x} = (x_1, \ldots, x_{b_2(M)})$ for $H^2(X, \mathbf{Z})$ and $\underline{y} = (y_1, \ldots, y_{b_4(M)})$ for $H^4(X, \mathbf{Z})$. Recall that by definition E-manifolds are oriented, so that the above data specify a basis for $H^*(X, \mathbf{Z})$, such that the bases for $H^i(X, \mathbf{Z})$ and $H^{8-i}(X, \mathbf{Z})$ are dual to each other with respect to the cup product. An *isomorphism between piecewise linear (smooth) based E-manifolds* $(X, \underline{x}, \underline{y})$ and $(X', \underline{x}', \underline{y}')$ is an orientation preserving piecewise linear (smooth) isomorphism $f \colon X \longrightarrow X'$ with $f^*(\underline{x}') = \underline{x}$

and $f^*(\underline{y}') = \underline{y}$. We denote by $\mathfrak{I}^{\mathrm{PL}(C^\infty)}(b, b')$ the set of isomorphy classes of piecewise linear (smooth) based E-manifolds $(X, \underline{x}, \underline{y})$ of dimension eight with vanishing second Stiefel-Whitney class, $b_2(X) = b$, and $b_4(X) = b'$.

2.1 The classical invariants

In the terminology of [24], the classical invariants of an E-manifold consist of its cohomology ring, the Stiefel-Whitney classes, the Wu classes, the Pontrjagin classes, the Euler class, the Steenrod squares, the reduced Steenrod powers, and the Pontrjagin powers. For an eight-dimensional E-manifold X with vanishing second Stiefel-Whitney class, the main result of [24] states that the classical invariants are fully determined by the following invariants:

1. The cup product map
$$\begin{aligned}\delta_X \colon S^2H^2(X,\mathbf{Z}) &\longrightarrow H^4(X,\mathbf{Z})\\ x\otimes x' &\longmapsto x\cup x'\,.\end{aligned}$$
2. The intersection form
$$\begin{aligned}\gamma_X \colon S^2H^4(X,\mathbf{Z}) &\longrightarrow \mathbf{Z}\\ y\otimes y' &\longmapsto (y\cup y')[X]\,.\end{aligned}$$
Here, $[X] \in H_8(X,\mathbf{Z})$ is the fundamental class determined by the orientation.
3. The first Pontrjagin class $p_1(X) \in H^4(X,\mathbf{Q})$.

Remark 2.1. The above invariants are not independent. By associativity of the cohomology ring, the following relation holds

$$\delta_X^*(\gamma_X) \in S^4H^2(X,\mathbf{Z})^\vee, \tag{1}$$

i.e.,

$$\gamma_X\big(\delta_X(x_1\otimes x_2)\otimes\delta_X(x_3\otimes x_4)\big) = \gamma_X\big(\delta_X(x_1\otimes x_3)\otimes\delta_X(x_2\otimes x_4)\big)\,,$$

for all $x_1, x_2, x_3, x_4 \in H^2(X,\mathbf{Z})$. Furthermore, PROPOSITION ([24], Prop. A.7 or Cor. 3.14 below). *For every element* $y \in H^4(X,\mathbf{Z})$ *we have*

$$p_1(X)y \equiv 2y^2 \mod 4 . \tag{2}$$

Note that this implies $p_1(X) \in H^4(X,\mathbf{Z})$. If, in addition, X is differentiable then for every integral lift $W \in H^2(X,\mathbf{Z})$ of $w_2(X)$ one has

$$3p_1(X)^2 - 14p_1(X)W^2 + 7W^4 \equiv 12 \operatorname{Sign}(\gamma_X) \mod 2688 . \tag{3}$$

Müller has also shown [24] that these relations imply all the other relations among the classical invariants of X. Conversely, a piecewise linear manifold X whose invariants obey relation (3) admits a differentiable structure [18], [24].

We are led to the following algebraic concept: A *system of invariants of type* (b,b') is a triple (δ,γ,p), consisting of

- a homomorphism $\delta\colon S^2\mathbf{Z}^{\oplus b} \longrightarrow \mathbf{Z}^{\oplus b'}$,
- a unimodular symmetric bilinear form $\gamma\colon S^2\mathbf{Z}^{\oplus b'} \longrightarrow \mathbf{Z}$, and
- an element $p \in \mathbf{Z}^{\oplus b'}$.

We denote by $Z(b,b')$ the set of systems of invariants of type (b,b').

Now, let $(X,\underline{x},\underline{y})$ be a based eight-dimensional E-manifold. This defines a set of invariants $Z_{(X,\underline{x},\underline{y})} := (\delta_X,\gamma_X,p_1(X))$ of type $(b_2(X),b_4(X))$. Thus, we have natural maps

$$\begin{aligned} Z^{\mathrm{PL}(C^\infty)}(b,b')\colon \mathfrak{J}^{\mathrm{PL}(C^\infty)}(b,b') &\longrightarrow Z(b,b') \\ [X,\underline{x},\underline{y}] &\longmapsto Z_{(X,\underline{x},\underline{y})} . \end{aligned}$$

It will be the concern of our paper to understand the maps $Z^{\mathrm{PL}(C^\infty)}$ as well as possible. The first result can be easily derived from Wall's work [36] and deals with the case $b=0$. It will be proved in detail in Section 4.1.

THEOREM 2.2. i) *The map* $Z^{\mathrm{PL}}(0,b')$ *is injective. Its image consists precisely of those elements which satisfy the relations* (1) *and* (2).

ii) *Given two smooth based E-manifolds* $(X,\underline{y})$ *and* $(X',\underline{y}')$ *with* $b_2(X)=0=b_2(X')$ *and* $Z_{(X,\underline{y})}=Z_{(X',\underline{y}')}$, *there exists an exotic* 8*-sphere* Σ *such that* $(X\#\Sigma,\underline{y})$ *and* $(X',\underline{y}')$ *are smoothly isomorphic. In particular, the fibres of* Z^{C^∞} *have cardinality at most two. The image of* Z^{C^∞} *consists exactly of those elements which satisfy the relations* (1), (2), *and* (3).

2.2 MANIFOLDS WITH TRIVIAL CUP FORM δ_X

In addition to describing the explicit geometric meaning of the system of invariants of an E-manifold X with $w_2(X) = 0$, we will describe those manifolds X for which the cup form δ_X vanishes.

For any $b > 0$, let $\mathcal{FC}_b^{\mathrm{PL}(\mathcal{C}^\infty)}$ be the group of isotopy classes of piecewise linear (smooth) embeddings of b disjoint copies of $S^5 \times D^3$ into S^8. The following result will be established in Section 3.7.

PROPOSITION 2.3. $\mathrm{FL}_b := \mathcal{FC}_b^{\mathcal{C}^\infty} \cong \mathcal{FC}_b^{\mathrm{PL}}$.

Given an element $[l] \in \mathrm{FL}_b$, we can perform surgery along the link l and get a smooth based E-manifold $(X(l), \underline{x}(l))$ with $w_2(X) = 0$ and $b_4(X) = 0$, which is well defined up to smooth isomorphy of based manifolds.

We will also use the following notation: Fix a pair $(\gamma, p) \in Z(0, b')$ which satisfies relation (2) (and (3)) and denote by $\mathfrak{I}^{\mathrm{PL}(\mathcal{C}^\infty)}(b, \gamma, p)$ the set of isomorphy classes of based piecewise linear (smooth) E-manifolds $(X, \underline{x}, \underline{y})$ with $w_2(X) = 0$, $b_2(X) = b$, $\gamma_X = \gamma$, and $p_1(X) = p$. Pick a three-connected piecewise linear (smooth) based E-manifold $(X_0, \underline{y}_0)$ with $\gamma_{X_0} = \gamma$ and $p_1(X_0) = p$. In the smooth case, let $\vartheta^8 \cong \mathbf{Z}_2$ [17] be the group of exotic 8-spheres, and set $\vartheta(X_0) := \vartheta^8$, if X_0 is not diffeomorphic to $X_0 \# \Sigma$, Σ a generator for ϑ^8, and $\vartheta(X_0) := \{[S^8]\} \subset \vartheta^8$ otherwise. Now, we define maps

$$K^{\mathrm{PL}}(b, \gamma, p)\colon\ \mathrm{FL}_b \longrightarrow \mathfrak{I}^{\mathrm{PL}}(b, \gamma, p)$$
$$[l] \longmapsto \left[X(l) \# X_0, \underline{x}(l), \underline{y}_0\right]$$

and

$$K_{X_0}^{\mathcal{C}^\infty}(b, \gamma, p)\colon\ \mathrm{FL}_b \oplus \vartheta(X_0) \longrightarrow \mathfrak{I}^{\mathcal{C}^\infty}(b, \gamma, p)$$
$$([l]\,, [\Sigma]) \longmapsto \left[X(l) \# X_0 \# \Sigma, \underline{x}(l), \underline{y}_0\right] .$$

In $\mathfrak{I}^{\mathrm{PL}(\mathcal{C}^\infty)}(b, \gamma, p)$, we mark the class $\left[\left(\#_{i=1}^b (S^2 \times S^6)\right) \# X_0, \underline{x}, \underline{y}_0\right]$, where $\underline{x}$ comes from the natural basis of $H^2(\#_{i=1}^b (S^2 \times S^6), \mathbf{Z})$. Then our main result is the following

THEOREM 2.4. i) *For every* $b > 0$ *and every pair* (γ, p) *which satisfies the relation* (2),

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & \mathrm{FL}_b & \stackrel{K^{\mathrm{PL}}(b,\gamma,p)}{\longrightarrow} & \mathfrak{J}^{\mathrm{PL}}(b,\gamma,\varphi) & \longrightarrow & \mathrm{Hom}(S^2\mathbf{Z}^b, Z^{b'}) \\ 1 & \longmapsto & [\textit{trivial link}] & & [X,\underline{x},\underline{y}] & \longmapsto & \delta_X \end{array}$$

is an exact sequence of pointed sets.

ii) *For every* $b > 0$ *and every pair* (γ, p) *which satisfies the relations* (2) *and* (3),

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & \mathrm{FL}_b \oplus \vartheta(X_0) & \stackrel{K^{C^\infty}_{X_0}(b,\gamma,p)}{\longrightarrow} & \mathfrak{J}^{C^\infty}(b,\gamma,\varphi) & \longrightarrow & \mathrm{Hom}(S^2\mathbf{Z}^b, Z^{b'}) \\ 1 & \longmapsto & [\textit{trivial link}] & & [X,\underline{x},\underline{y}] & \longmapsto & \delta_X \end{array}$$

is an exact sequence of pointed sets.

The proof will be given in Sections 4.2 and 4.3.

REMARK 2.5. i) In the PL setting, we will show that the inclusion of FL_b into $\mathfrak{J}^{\mathrm{PL}}(b,\gamma,\varphi)$ extends to an action of FL_b on $\mathfrak{J}^{\mathrm{PL}}(b,\gamma,\varphi)$, such that the orbits are the fibres of the map $[X,\underline{x},\underline{y}] \longmapsto \delta_X$.

ii) On all the sets occurring in Theorem 2.4 there are natural $(\mathrm{GL}_b(\mathbf{Z}) \times \mathrm{GL}_{b'}(\mathbf{Z}))$-actions, and the maps are equivariant for these actions. Therefore, by forming the equivalence classes with respect to these actions, we get the classification of E-manifolds with vanishing second Stiefel-Whitney class up to orientation preserving piecewise linear (smooth) isomorphy.

iii) We will discuss in Section 3.7 the structure of the group FL_b. It turns out that it is finite if and only if $b = 1$. It follows easily that the set of $\mathrm{GL}_b(\mathbf{Z})$-equivalence classes in FL_b is infinite for $b \geq 2$. Thus, the cohomology ring and the characteristic classes classify E-manifolds of dimension eight up to finite indeterminacy only if the second Betti number is at most one.

The starting point of our proof of the above results will be the theory of minimal handle decompositions of Smale which states that X can be obtained from D^8 by first attaching $b_2(X)$ 2-handles, then $b_4(X)$ 4-handles, then $b_2(X)$ 6-handles and finally one 8-handle. At each step, the attachment will be determined by the isotopy class of a certain framed link in a 7-manifold, and we will first explain how to read off the isotopy class of the attaching links for the 2- and 4-handles from the invariants.