3. Preliminaries

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3. Preliminaries

We collect in this paragraph the background material and some preliminary results which we will use in our proof. Most of the results are by now standard results from various parts of algebraic, differential, and piecewise linear topology.

3.1 The structure of manifolds: handle attachment and surgery

Let M be an m-dimensional manifold with boundary. Suppose we are given an embedding $f\colon S^{\lambda-1} \times D^{m-\lambda} \longrightarrow \partial M$. We then define

$$M' := M \cup_f \left(D^\lambda \times D^{m-\lambda}\right)$$

and say that M' is obtained from M by the *attachment of a λ-handle along f*. Moreover, $f(S^{\lambda-1} \times \{0\})$ is called the *attaching sphere*, $D^\lambda \times \{0\}$ the *core disc*, and $\{0\} \times S^{m-\lambda-1}$ the *belt sphere.* We will often simply write $M' = M \cup H^\lambda$.

Remark 3.1. i) If M is assumed to be differentiable and f to be a differentiable embedding, handle attachment can be described in such a way that the resulting manifold is again differentiable (see [17], VI, §§6 and 8), i.e., no "smoothing of the corners" is required.

ii) If M is oriented, then M' will inherit an orientation which is compatible with the given orientation of M' and the natural orientation of $D^\lambda \times D^{m-\lambda}$, if and only if f *reverses* the orientations.

The next operation we consider was introduced by Milnor [21] and Wallace [38] and goes back to Thom. For this, let N be a manifold of dimension $m-1$ and $f\colon S^{\lambda-1} \times D^{m-\lambda} \longrightarrow N$ an embedding. Denote by $\bar{f}$ the restriction of f to $S^{\lambda-1} \times S^{m-\lambda-1}$, and set

$$\chi(N,f) := \left(N \setminus \operatorname{int} f(S^{\lambda-1} \times D^{m-\lambda})\right) \cup_{\bar{f}} \left(D^\lambda \times S^{m-\lambda-1}\right).$$

We say that $\chi(N,f)$ is constructed from N by *surgery along f*. Informally speaking, we remove from N a $(\lambda-1)$-sphere with trivial normal bundle and replace it with an $(m-\lambda-1)$-sphere, again with trivial normal bundle.

Remark 3.2. i) If N is oriented, then f has to be orientation preserving for $\chi(N,f)$ to inherit a natural orientation from those of N and $D^\lambda \times S^{m-\lambda-1}$. This is because $S^{\lambda-1} \times S^{m-\lambda-1}$ inherits the reversed orientation as boundary of $N \setminus \operatorname{int} f(S^{\lambda-1} \times D^{m-\lambda})$.

ii) The operations of handle attachment and surgery are closely related: Let M be an m-dimensional manifold with boundary $N := \partial M$ and $f\colon S^{\lambda-1} \times D^{m-\lambda} \longrightarrow N$ an embedding. Now, attach a λ-handle along f in order to obtain M'. Then $\partial M' = \chi(N,f)$.

We will also perform a "surgery in pairs". For this, N is assumed to be an $(m-1)$-dimensional manifold, and K a submanifold of dimension $k-1$. Assume that, for some $\lambda \leq k$, we are given an embedding $f\colon S^{\lambda-1}\times D^{m-\lambda} \to N$ which induces an embedding $f^* := f|_{S^{\lambda-1}\times D^{k-\lambda}}\colon S^{\lambda-1} \times D^{k-\lambda} \longrightarrow K$. Then $\chi(K,f^*)$ is naturally contained as a submanifold in $\chi(N,f)$.

The next result is a special case of the "minimal presentation theorem" of Smale [31] and is crucial for the explicit analysis of the structure of a manifold.

THEOREM 3.3. *Let X be a closed simply connected manifold of dimension $m \geq 6$ with torsion free homology. Then there exists a sequence of submanifolds*

$$D^m \cong W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_m = X,$$

such that W_i is obtained from W_{i-1} by attaching $b_i(X)$ i-handles, $i = 1,\ldots,m$.

Moreover, for any such sequence, there exists a dual sequence

$$\overline{W}_0 \subset \overline{W}_1 \subset \cdots \subset \overline{W}_m = X,$$

such that the attaching $(i-1)$-spheres in $\overline{W}_{i-1}$ coincide with the belt spheres in W_{m-i}, $i = 1,\ldots,m$.

Proof. For differentiable manifolds, an attractive presentation of the relevant material is contained in Chapters VII and VIII of [17]. In the piecewise linear category, handle decompositions are discussed in [27] (cf. also [13]). However, the statement concerning the number of handles is not explicitly given there. Nevertheless, one verifies that the necessary tools (such as Whitney lemma and handle sliding) are also proved in [27]. □

REMARK 3.4. i) Retracting all λ-handles to their core discs, starting with $\lambda = 0$, yields a CW-complex which is homotopy equivalent to X (cf. [27], p. 83).

ii) Observe that, by i), a handle decomposition as in Theorem 3.3 yields a *preferred basis* for $H_*(X,\mathbf{Z})$. By renumbering, orientation reversal in the attaching spheres, and handle sliding, one can obtain any basis of $H_*(X,\mathbf{Z})$ as the preferred basis of a handle decomposition ([17], (1.7), p. 148).

iii) If X comes with an orientation, we may assume that D^m is orientation preservingly embedded and that all attaching maps are orientation reversing.

3.2 Consequences for E-manifolds of dimension eight with $w_2 = 0$

Let X be a piecewise linear (smooth) E-manifold of dimension eight with $w_2(X) = 0$. The first observation concerns the structure of W_2.

LEMMA 3.5. *One has* $W_2 \cong \#_{i=1}^b (S^2 \times D^6)$.

Proof. The manifold W_2 is an $(8,1)$-handle body and as such homeomorphic to the boundary connected sum of b D^6-bundles over S^2 ([17], § 11, p. 115). As $\pi_1(\mathrm{SO}(4)) \cong \mathbf{Z}_2$ and we have requested $w_2(X) = 0$, the claim follows. □

The next consequence is

The manifold W_4 is determined by a framed link of $b_4(X)$ three-dimensional spheres in $\#_{i=1}^b (S^2 \times S^5)$.

We shall look into the classification of such links below. The third consequence is

LEMMA 3.6. i) $\partial W_4 \cong \#_{i=1}^b (S^2 \times S^5)$.

ii) *The manifold X is of the form* $W_4 \cup_f \#_{i=1}^b (S^2 \times D^6)$ *where*

$$f \colon \#_{i=1}^b (S^2 \times S^5) \longrightarrow \partial W_4$$

is a piecewise linear (smooth) isomorphism, such that f_* maps the canonical basis of $H_2(\#_{i=1}^b (S^2 \times S^5), \mathbf{Z})$ *to the preferred basis of* $H_2(\partial W^4, \mathbf{Z})$.

Proof. i) This follows because $\partial W_4 \cong \partial \overline{W}_2$. ii) follows because $X = W_4 \cup \overline{W}_2$, and $\overline{W}_2 \cong \#_{i=1}^b (S^2 \times D^6)$, by Lemma 3.5. □

3.3 Homotopy vs. isotopy

By Theorem 3.3, the manifold is determined by the ambient isotopy classes of the attaching maps. However, the topological invariants of the manifold give us at best their homotopy classes. It is, therefore, important to have theorems granting that this is enough. In the setting of differentiable manifolds, we have

THEOREM 3.7 (Haefliger [6], [7]). *Let S be a closed differentiable manifold of dimension n and M an m-dimensional differentiable manifold without boundary. Let $f\colon S \longrightarrow M$ be a continuous map and $k \geq 0$, such that*

$$\pi_i(f)\colon \pi_i(S) \longrightarrow \pi_i(M)$$

is an isomorphism for $0 \leq i \leq k$ and surjective for $i = k+1$. Then the following is satisfied:

1. *If $m \geq 2n - k$ and $n > 2k + 2$, then f is homotopic to a differentiable embedding.*
2. *If $m > 2n - k$ and $n \geq 2k + 2$, then two differentiable embeddings of S into M which are homotopic are also ambient isotopic.*

In the setting of piecewise linear manifolds, similar results hold true. We refer to Haefliger's survey article [9]. For our purposes, the result stated below will be sufficient.

THEOREM 3.8. *Suppose S is a closed n-dimensional manifold, M an m-dimensional manifold without boundary, and $f\colon S \longrightarrow M$ a continuous map. Assume one has*

- *$m - n \geq 3$;*
- *S is $(2n - m + 1)$-connected;*
- *M is $(2n - m + 2)$-connected.*

Then:

(1) *f is homotopic to an embedding;*

(2) *two embeddings which are homotopic to f are ambient isotopic.*

Proof. The theorem of Irwin ([27], Thm. 7.12 and Ex. 7.14, [13], Thm. 8.1) yields (1) and that f_1 and f_2 as in (2) are concordant. But, since $m - n \geq 3$, concordance implies ambient isotopy ([13], Chap. IX). □

COROLLARY 3.9. *Let $S := S^3$ and M a simply connected differentiable or piecewise linear manifold of dimension 7 without boundary. Then $\pi_3(M)$ classifies differentiable and piecewise linear embeddings, respectively, of S^3 into M up to ambient isotopy.*

COROLLARY 3.10 (Zeeman's unknotting theorem [39]). *For every m, n with $m - n \geq 3$, any piecewise linear embedding of S^n into S^m is isotopic to the standard embedding.*

3.4 Some 4-dimensional CW-complexes

By Remark 3.4, a handle decomposition of X gives us a CW-complex which is homotopy equivalent to X. The following discussion will enable us to understand the 4-skeleton of that complex.

Let $W := S^2 \vee \cdots \vee S^2$ be the b-fold wedge product of 2-spheres. Suppose X is the CW-complex obtained by attaching a 4-cell to W via the map $g \in \pi_3(W)$. The Hilton-Milnor theorem ([30], Thm. 7.9.4) asserts that

$$\pi_3(W) = \bigoplus_{i=1}^{b} \pi_3(S^2) \oplus \bigoplus_{1 \leq i < j \leq b} \pi_3(S^3).$$

Choosing the standard generators for $\pi_3(S^2)$ and $\pi_3(S^3)$, we can describe g by a tuple $(l_i, i = 1, \ldots, b; l_{ij}, 1 \leq i < j \leq b)$ of integers. These integers determine the cohomology ring of $X = W \cup_g D^4$ as follows:

PROPOSITION 3.11. *Let $y \in H^4(X, \mathbf{Z})$ be the generator of $H^4(X, \mathbf{Z})$ given by the attached 4-cell and $x_1, \ldots, x_b$ the canonical basis of $H^2(X, \mathbf{Z}) = H^2(W, \mathbf{Z})$. Then*

$$\begin{aligned} x_i \cup x_j &= l_{ij} \cdot y, \qquad && 1 \leq i < j \leq b, \\ x_i \cup x_i &= l_i \;\cdot y, && i = 1, \ldots, b. \end{aligned}$$

This is proved like [22], (1.5), p. 103. We recall the proof in the following example.

EXAMPLE 3.12. We treat the case $b = 2$. Consider the embedding

$$\iota \colon S^2 \vee S^2 \hookrightarrow S^2 \times S^2 \hookrightarrow \mathbf{CP}^\infty \times \mathbf{CP}^\infty.$$

The standard basis for $H^4(\mathbf{CP}^\infty \times \mathbf{CP}^\infty, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 3}$ is given by the elements y_1, y_2, y_3 obtained from attaching D^4 via $(1,0;0)$, $(0,0;1)$, and $(0,1;0)$, respectively. Let $h \colon D^4 \longrightarrow D^4 \vee D^4 \vee D^4$ be the canonical map followed by

$$\big(\vartheta \cdot x \longmapsto \vartheta \cdot m_{l_1}(x)\big) \vee \big(\vartheta \cdot x \longmapsto \vartheta \cdot m_{l_{12}}(x)\big) \vee \big(\vartheta \cdot x \longmapsto \vartheta \cdot m_{l_2}(x)\big).$$

Here, m_k stands for a representative of $[k \cdot \mathrm{id}_{S^3}] \in \pi_3(S^3)$ and $D^4 = \big\{\vartheta \cdot x \mid x \in S^3, \vartheta \in [0,1]\big\}$. Now, h and ι glue to a map $f \colon X \longrightarrow \mathbf{CP}^\infty \times \mathbf{CP}^\infty$, and

$$\begin{aligned} f^* \colon H^4(\mathbf{CP}^\infty \times \mathbf{CP}^\infty, \mathbf{Z}) &\longrightarrow H^4(X, \mathbf{Z}) \\ a_1 y_1 + a_2 y_2 + a_3 y_3 &\longmapsto \big(a_1 l_1 + a_2 l_{12} + a_3 l_2\big) y, \end{aligned}$$

so that the assertion follows from the naturality of the cup-product.

3.5 Pontrjagin classes and $\pi_3(\mathrm{SO}(4))$

Vector bundles of rank 4 over S^4 are classified by elements in $\pi_3(\mathrm{SO}(4))$. In our setting, such vector bundles will appear as normal bundles. We recall, therefore, the description of that group and relate it to Pontrjagin classes and self intersection numbers.

First, look at the natural map $\pi_3(\mathrm{SO}(4)) \longrightarrow \pi_3\big(\mathrm{SO}(4)/\mathrm{SO}(3)\big) = \pi_3(S^3)$. This map has a splitting ([32], § 22.6) which induces an isomorphism

$$\pi_3(\mathrm{SO}(4)) = \pi_3(\mathrm{SO}(3)) \oplus \pi_3(S^3).$$

Let α_3 be the generator for $\pi_3(\mathrm{SO}(3)) \cong \mathbf{Z}$ from [32], § 22.3, and $\beta_3 := [\mathrm{id}_{S^3}] \in \pi_3(S^3)$, so that we obtain the isomorphism $\mathbf{Z} \oplus \mathbf{Z} \longrightarrow \pi_3(\mathrm{SO}(4))$, $(k_1, k_2) \longmapsto k_1\alpha_3 + k_2\beta_3$. Finally, the kernel of the map $\pi_3(\mathrm{SO}(4)) \longrightarrow \pi_3(\mathrm{SO})$ to the stable homotopy group is generated by $-\alpha_3 + 2\beta_3$ ([32], § 23.6), whence [23], (20.9), implies

Proposition 3.13. *Let E be the vector bundle over S^4 defined by the element $k_1\alpha_3 + k_2\beta_3 \in \pi_3(\mathrm{SO}(4))$. Then*

$$p_1(E) = \pm(2k_1 + 4k_2).$$

Corollary 3.14. *Let $f\colon S^4 \longrightarrow M$ be a differentiable embedding of S^4 into the differentiable 8-manifold M. Let $E := f^*T_M/T_{S^4}$ be the normal bundle. Then the self intersection number s of $f(S^4)$ in M satisfies*

$$2s \equiv p_1(E) \mod 4.$$

Proof. If E is given by the element $k_1\alpha_3 + k_2\beta_3 \in \pi_3(\mathrm{SO}(4))$, then $s = k_2$ ([17], (5.4), p. 72). Since $p_1(E) = \pm(2k_2 + 4k_1)$, the claim follows. $\square$

3.6 Links of 3-spheres in $\#_{i=1}^b(S^2 \times S^5)$

If X is a closed E-manifold of dimension 8 with $w_2(X) = 0$, then $W_2 := \#_{i=1}^b(S^2 \times D^6)$, $b = b_2(X)$, by Lemma 3.5. Thus, W_4 is determined by a framed link of 3-spheres in $\partial W_2 = \#_{i=1}^b(S^2 \times S^5)$. Therefore, we will now classify such links.

So, let $W := \#_{i=1}^b(S^2 \times S^5)$ be a b-fold connected sum. We can choose b disjoint 2-spheres S_i^2, $i = 1, \ldots, b$, embedded in W and representing the natural basis of $H_2(W, \mathbf{Z})$. One checks that the homotopy type of W is given up to dimension 4 by the b-fold wedge product $S^2 \vee \cdots \vee S^2$. Suppose we are given a link of b' three-dimensional spheres, i.e., we are given b' differentiable embeddings $g_i\colon S^3 \longrightarrow W$, $i = 1, \ldots, b'$, with $g_i(S^3) \cap g_j(S^3) = \varnothing$ for $i \neq j$.

By the transversality theorem ([17], IV.(2.4)), one sees that we may assume $S_i^2 \cap g_j(S^3) = \varnothing$ for all i and j.

By Corollary 3.9, the ambient isotopy class of the embedding g_k is determined by the element $\varphi_k := [g_k] \in \pi_3(W_k)$, $W_k := W \setminus \bigcup_{j \neq k} g_j(S^3)$, $k = 1, \dots, b'$. We clearly have (compare [8])

$$\pi_3(W_k) = \pi_3\big(\underbrace{S^2 \vee \cdots \vee S^2}_{b\times} \vee \underbrace{S^3 \vee \cdots \vee S^3}_{(b'-1)\times}\big)\,,$$

so that the Hilton-Milnor theorem yields

$$\pi_3(W_k) = \bigoplus_{i=1}^{b} \pi_3(S^2) \oplus \bigoplus_{1 \le i < j \le b} \pi_3(S^3) \oplus \bigoplus_{j \neq k} \pi_3(S^3)\,.$$

Hence, we write φ_k as a tuple of integers:

$$\varphi_k = \big(l_i^k,\ i = 1, \dots, b;\quad l_{ij}^k,\ 1 \le i < j \le b;\ \lambda_{kj},\ j \neq k\big)\,.$$

Observe that, for $j \neq k$, φ_k is mapped under the natural homomorphism

$$\pi_3(W_k) \longrightarrow H_3(W_k, \mathbf{Z}) \longrightarrow H_3\big(W \setminus g_j(S^3), \mathbf{Z}\big)\ (\cong \mathbf{Z})$$

to the image of the fundamental class of S^3 under g_{j*}. Thus, λ_{kj} is just the 'usual' linking number of the spheres $g_k(S^3)$ and $g_j(S^3)$ in W (compare [8]).

3.7 Links of 5-spheres in S^8

Let $\mathcal{FC}_b^{\mathrm{PL}(\mathcal{C}^\infty)}$ be as before, and let $\mathcal{C}_b^{\mathrm{PL}(\mathcal{C}^\infty)}$ be the group of isotopy classes of piecewise linear (smooth) embeddings of b disjoint copies of S^5 into S^8. For $b = 1$, these groups are studied in [10], [19], and [20]. A brief summary with references of results in the case $b > 1$ is contained in Section 2.6 of [11]. We will review some of this material below.

Proposition 3.15. *We have $\mathcal{FC}_1^{\mathcal{C}^\infty} \cong \mathcal{FC}_1^{\mathrm{PL}} \cong \mathbf{Z}_2$.*

Proof. Since $\pi_5(\mathrm{SO}(3)) \cong \mathbf{Z}_2$, the standard embedding of S^5 into S^8 with its two possible framings provides an injection of $\mathbf{Z}_2$ into $\mathcal{FC}_1^{\mathrm{PL}(\mathcal{C}^\infty)}$. By Zeeman's unknotting theorem 3.10, the map $\mathbf{Z}_2 \longrightarrow \mathcal{FC}_1^{\mathrm{PL}}$ is an isomorphism. As remarked in Section 2.6 of [11], $\mathcal{FC}_1^{\mathrm{PL}}$ is isomorphic to $\mathcal{F}\vartheta$, the group of h-cobordism classes of framed submanifolds of S^8 which are homotopy 5-spheres. Moreover, by [10] and [19], there is an exact sequence

$$\cdots \longrightarrow \vartheta^6 \longrightarrow \mathcal{FC}_1^{\mathcal{C}^\infty} \longrightarrow \mathcal{F}\vartheta \longrightarrow \vartheta^5 \longrightarrow \cdots\,.$$

As the groups ϑ^5 and ϑ^6 of exotic 5- and 6-spheres are trivial [17], our claim is settled. $\square$

Let $\mathrm{L}_b \subset \mathcal{C}_b^{C^\infty}$ be the subgroup of those embeddings for which the restriction to each component is isotopic to the standard embedding. As observed in Section 2.6 of [11], Zeeman's unknotting theorem 3.10 implies that $\mathrm{L}_b = \mathcal{C}_b^{\mathrm{PL}}$. The following result settles Proposition 2.3:

COROLLARY 3.16. $\mathcal{F}\mathcal{C}_b^{C^\infty} \cong \mathcal{F}\mathcal{C}_b^{\mathrm{PL}} \cong \mathrm{L}_b \oplus \bigoplus_{i=1}^{b} \mathbf{Z}_2\,.$

For the group L_b, Theorem 1.3 of [11] provides a fairly explicit description as an extension of abelian groups. For this, consider the b-fold wedge product $\bigvee_{i=1}^{b} S^2$ of 2-spheres together with its inclusion $i\colon \bigvee_{i=1}^{b} S^2 \hookrightarrow \times_{i=1}^{b} S^2$ into the b-fold product of 2-spheres. Finally, let $p_i\colon \bigvee_{i=1}^{b} S^2 \longrightarrow S^2$ be the projection onto the i^{th} factor, $i = 1, \dots, b$. Set, for $m = 1, 2, \dots$,

$$\Lambda_{b,j}^m := \mathrm{Ker}\big(\pi_m(p_j)\colon \pi_m(\bigvee_{i=1}^{b} S^2) \longrightarrow \pi_m(S^2)\big), \quad j = 1, \dots, b,$$

$$\Lambda_b^m := \bigoplus_{j=1}^{b} \Lambda_{b,j}^m$$

and

$$\Pi_b^m := \mathrm{Ker}\big(\pi_m(i)\colon \pi_m(\bigvee_{i=1}^{b} S^2) \longrightarrow \bigoplus_{i=1}^{b} \pi_m(S^2)\big)\,,$$

and define

$$w_b^m\colon \Lambda_b^m \longrightarrow \Pi_b^{m+1}$$

on $\Lambda_{b,j}^m$ by $w_b^m(\alpha) := [\alpha, \iota_i]$. Here, $[.,.]$ stands for the Whitehead product inside the homotopy groups of $\bigvee_{i=1}^{b} S^2$ and $\iota_i\colon S^2 \hookrightarrow \bigvee_{i=1}^{b} S^2$ for the inclusion of the i^{th} factor, $i = 1, \dots, b$. Theorem 1.3 of [11] yields in our situation

THEOREM 3.17. *There is an exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Coker}(w_b^6) \longrightarrow \mathrm{L}_b \longrightarrow \mathrm{Ker}(w_b^5) \longrightarrow 0\,.$$

We remark that the formulas of Steer [33] might be used for the explicit computation of Whitehead products and thus for the determination of $\mathrm{Coker}(w_b^6)$ and $\mathrm{Ker}(w_b^5)$. The free part of L_b, e.g., can be obtained quite easily. We confine ourselves to prove the following important fact.

COROLLARY 3.18. *The group* L_b *has positive rank for* $b \geq 2$.

Proof. Let $\mathbf{L}_b := \bigoplus_{l\geq 1} \mathbf{L}_{b,l}$ be the free graded Lie algebra with $\mathbf{L}_{b,1} := \bigoplus_{i=1}^{b} \mathbf{Z} \cdot e_i$. For $l = 2,3,\ldots$, let $e_1^l,\ldots,e_{d_l}^l$ be a basis for $\mathbf{L}_{b,l}$ consisting of iterated commutators of the e_i. By assigning ι_i to e_i, every iterated commutator $c \in \mathbf{L}_{b,l}$ in the e_i defines an element $\alpha(c) \in \pi_{l+1}(\bigvee_{i=1}^{b} S^2)$.

To settle our claim, it is certainly sufficient to show that $\mathrm{Coker}(w_b^6)$ has positive rank. Now, by the Hilton-Milnor theorem

$$\Pi_b^7 \cong \bigoplus_{l=3}^{7}\bigoplus_{k=1}^{d_{l-1}} \pi_7(S^l) \cdot \alpha(e_k^{l-1}).$$

Note that $\pi_7(S^l)$ is finite for $l \notin \{4,7\}$ (see [32] and [35] for the explicit description of those groups). The Hopf fibration $S^7 \longrightarrow S^4$ [32], on the other hand, yields a decomposition $\pi_7(S^4) \cong \pi_6(S^3)\oplus\pi_7(S^7) \cong \mathbf{Z}_{12}\oplus\mathbf{Z}$. Therefore, it will suffice to show that the free part of Λ_b^6 is mapped to $\bigoplus_{j=1}^{d_6} \pi_7(S^7)\cdot\alpha(e_j^6)$. For $j = 1,\ldots,b$, we have

$$\Lambda_{b,j}^6 \cong \bigoplus_{i\neq j} \pi_6(S^2) \cdot \iota_i \oplus \bigoplus_{l=3}^{6}\bigoplus_{k=1}^{d_{l-1}} \pi_6(S^l) \cdot \alpha(e_k^{l-1}).$$

The group $\pi_6(S^l)$ is finite for $l < 6$, and we obviously have $[\alpha(e_k^5),\iota_j] = \alpha([e_k^5,e_j])$. If we expand the commutator $[e_k^5,e_j]$ in the basis $e_1^6,\ldots,e_{d_6}^6$, we find an expansion for $[\alpha(e_k^6),\iota_j]$ in terms of the $\alpha(e_k^6)$. □

COROLLARY 3.19. *The set of* $\mathrm{GL}_b(\mathbf{Z})$*-equivalence classes of elements in* L_b *is infinite for* $b \geq 2$.

Proof. We have seen that the $\mathrm{GL}_b(\mathbf{Z})$-set $\mathbf{L}_{b,3}$ is contained in the $\mathrm{GL}_b(\mathbf{Z})$-set L_b. The $\mathrm{GL}_b(\mathbf{Z})$-action on $\mathbf{L}_{b,3}$ originates from a homomorphism $\mathrm{GL}_b(\mathbf{Z}) \longrightarrow \mathrm{GL}(\mathbf{L}_{b,3}) := \mathrm{Aut}_{\mathbf{Z}}(\mathbf{L}_{b,3})$. In particular, any matrix $g \in \mathrm{GL}_b(\mathbf{Z})$ preserves the absolute value of the determinant of any d_3 elements in $\mathbf{L}_{b,3}$. This implies, for instance, that $a \cdot e_1^3$ and $b \cdot e_1^3$ cannot lie in the same $\mathrm{GL}_b(\mathbf{Z})$-orbit, if $0 \leq a < b$. □