3.1 The structure of manifolds: handle attachment and surgery

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3. Preliminaries

We collect in this paragraph the background material and some preliminary results which we will use in our proof. Most of the results are by now standard results from various parts of algebraic, differential, and piecewise linear topology.

3.1 The structure of manifolds: handle attachment and surgery

Let M be an m-dimensional manifold with boundary. Suppose we are given an embedding $f\colon S^{\lambda-1} \times D^{m-\lambda} \longrightarrow \partial M$. We then define

$$M' := M \cup_f \left(D^\lambda \times D^{m-\lambda}\right)$$

and say that M' is obtained from M by the *attachment of a λ-handle along f*. Moreover, $f(S^{\lambda-1} \times \{0\})$ is called the *attaching sphere*, $D^\lambda \times \{0\}$ the *core disc*, and $\{0\} \times S^{m-\lambda-1}$ the *belt sphere*. We will often simply write $M' = M \cup H^\lambda$.

REMARK 3.1. i) If M is assumed to be differentiable and f to be a differentiable embedding, handle attachment can be described in such a way that the resulting manifold is again differentiable (see [17], VI, §§6 and 8), i.e., no "smoothing of the corners" is required.

ii) If M is oriented, then M' will inherit an orientation which is compatible with the given orientation of M' and the natural orientation of $D^\lambda \times D^{m-\lambda}$, if and only if f *reverses* the orientations.

The next operation we consider was introduced by Milnor [21] and Wallace [38] and goes back to Thom. For this, let N be a manifold of dimension $m-1$ and $f\colon S^{\lambda-1} \times D^{m-\lambda} \longrightarrow N$ an embedding. Denote by $\bar{f}$ the restriction of f to $S^{\lambda-1} \times S^{m-\lambda-1}$, and set

$$\chi(N,f) := \left(N \setminus \operatorname{int} f(S^{\lambda-1} \times D^{m-\lambda})\right) \cup_{\bar{f}} \left(D^\lambda \times S^{m-\lambda-1}\right).$$

We say that $\chi(N,f)$ is constructed from N by *surgery along f*. Informally speaking, we remove from N a $(\lambda-1)$-sphere with trivial normal bundle and replace it with an $(m-\lambda-1)$-sphere, again with trivial normal bundle.

REMARK 3.2. i) If N is oriented, then f has to be orientation preserving for $\chi(N,f)$ to inherit a natural orientation from those of N and $D^\lambda \times S^{m-\lambda-1}$. This is because $S^{\lambda-1} \times S^{m-\lambda-1}$ inherits the reversed orientation as boundary of $N \setminus \operatorname{int} f(S^{\lambda-1} \times D^{m-\lambda})$.

ii) The operations of handle attachment and surgery are closely related: Let M be an m-dimensional manifold with boundary $N := \partial M$ and $f\colon S^{\lambda-1} \times D^{m-\lambda} \longrightarrow N$ an embedding. Now, attach a λ-handle along f in order to obtain M'. Then $\partial M' = \chi(N,f)$.

We will also perform a "surgery in pairs". For this, N is assumed to be an $(m-1)$-dimensional manifold, and K a submanifold of dimension $k-1$. Assume that, for some $\lambda \leq k$, we are given an embedding $f\colon S^{\lambda-1}\times D^{m-\lambda} \to N$ which induces an embedding $f^* := f|_{S^{\lambda-1}\times D^{k-\lambda}} : S^{\lambda-1} \times D^{k-\lambda} \longrightarrow K$. Then $\chi(K,f^*)$ is naturally contained as a submanifold in $\chi(N,f)$.

The next result is a special case of the "minimal presentation theorem" of Smale [31] and is crucial for the explicit analysis of the structure of a manifold.

THEOREM 3.3. *Let X be a closed simply connected manifold of dimension $m \geq 6$ with torsion free homology. Then there exists a sequence of submanifolds*

$$D^m \cong W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_m = X,$$

such that W_i is obtained from W_{i-1} by attaching $b_i(X)$ i-handles, $i = 1,\dots,m$.

Moreover, for any such sequence, there exists a dual sequence

$$\overline{W}_0 \subset \overline{W}_1 \subset \cdots \subset \overline{W}_m = X,$$

such that the attaching $(i-1)$-spheres in $\overline{W}_{i-1}$ coincide with the belt spheres in W_{m-i}, $i = 1,\dots,m$.

Proof. For differentiable manifolds, an attractive presentation of the relevant material is contained in Chapters VII and VIII of [17]. In the piecewise linear category, handle decompositions are discussed in [27] (cf. also [13]). However, the statement concerning the number of handles is not explicitly given there. Nevertheless, one verifies that the necessary tools (such as Whitney lemma and handle sliding) are also proved in [27]. □

REMARK 3.4. i) Retracting all λ-handles to their core discs, starting with $\lambda = 0$, yields a CW-complex which is homotopy equivalent to X (cf. [27], p. 83).

ii) Observe that, by i), a handle decomposition as in Theorem 3.3 yields a *preferred basis* for $H_*(X,\mathbf{Z})$. By renumbering, orientation reversal in the attaching spheres, and handle sliding, one can obtain any basis of $H_*(X,\mathbf{Z})$ as the preferred basis of a handle decomposition ([17], (1.7), p. 148).

iii) If X comes with an orientation, we may assume that D^m is orientation preservingly embedded and that all attaching maps are orientation reversing.

3.2 Consequences for E-manifolds of dimension eight with $w_2 = 0$

Let X be a piecewise linear (smooth) E-manifold of dimension eight with $w_2(X) = 0$. The first observation concerns the structure of W_2.

Lemma 3.5. *One has* $W_2 \cong \#_{i=1}^b (S^2 \times D^6)$.

Proof. The manifold W_2 is an $(8,1)$-handle body and as such homeomorphic to the boundary connected sum of b D^6-bundles over S^2 ([17], § 11, p. 115). As $\pi_1(\mathrm{SO}(4)) \cong \mathbf{Z}_2$ and we have requested $w_2(X) = 0$, the claim follows. □

The next consequence is

The manifold W_4 is determined by a framed link of $b_4(X)$ three-dimensional spheres in $\#_{i=1}^b (S^2 \times S^5)$.

We shall look into the classification of such links below. The third consequence is

Lemma 3.6. i) $\partial W_4 \cong \#_{i=1}^b (S^2 \times S^5)$.

ii) *The manifold X is of the form* $W_4 \cup_f \#_{i=1}^b (S^2 \times D^6)$ *where*

$$f\colon \#_{i=1}^b (S^2 \times S^5) \longrightarrow \partial W_4$$

is a piecewise linear (smooth) isomorphism, such that f_* maps the canonical basis of $H_2(\#_{i=1}^b (S^2 \times S^5), \mathbf{Z})$ to the preferred basis of $H_2(\partial W^4, \mathbf{Z})$.

Proof. i) This follows because $\partial W_4 \cong \partial \overline{W}_2$. ii) follows because $X = W_4 \cup \overline{W}_2$, and $\overline{W}_2 \cong \#_{i=1}^b (S^2 \times D^6)$, by Lemma 3.5. □

3.3 Homotopy vs. isotopy

By Theorem 3.3, the manifold is determined by the ambient isotopy classes of the attaching maps. However, the topological invariants of the manifold give us at best their homotopy classes. It is, therefore, important to have theorems granting that this is enough. In the setting of differentiable manifolds, we have