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**Artikel:** ON THE CLASSIFICATION OF CERTAIN PIECEWISE LINEAR AND DIFFERENTIABLE MANIFOLDS IN DIMENSION EIGHT AND AUTOMORPHISMS OF  $\sharp_{i=1}^b (S^2 \times S^5)$   
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## 4. PROOF OF THEOREM 2.2 AND THEOREM 2.4

From now on,  $X$  stands for an eight-dimensional E-manifold with  $w_2(X) = 0$ .

## 4.1 PROOF OF THEOREM 2.2

The classification result for 3-connected E-manifolds of dimension eight is a special case of a result of Wall's [36] and can be easily obtained with the methods described in [17], VII, §12. Let us recall the details, because we will need them later on.

We fix a basis  $\underline{b}$  for  $H_4(X, \mathbf{Z})$  and let  $\underline{y}$  be the dual basis of  $H^4(X, \mathbf{Z})$ . Then there is a handle presentation  $X = D^8 \cup H_1^4 \cup \dots \cup H_{b'}^4 \cup D^8$  with  $\underline{b}$  as the preferred basis. The manifold  $T := D^8 \cup H_1^4 \cup \dots \cup H_{b'}^4$  is determined by the ambient isotopy class of a framed link of 3-spheres in  $S^7$ , having  $b'$  components. Let us first look at such a link, forgetting the framing, i.e., suppose we are given embeddings  $g_i: S^3 \rightarrow S^7$  with  $S_i \cap S_j = \emptyset$  for  $i \neq j$ ,  $S_i := g_i(S^3)$ ,  $i = 1, \dots, b'$ . By 3.9, we may assume that the  $g_i$  are differentiable. Observe that the normal bundles of the  $S_i$  are trivial.

We equip  $S_i$  with the orientation induced via  $g_i$  by the standard orientation of  $S^3$  and the normal bundle of  $S_i$  with the orientation which is determined by requiring that the orientation of  $S_i$  followed by that of its normal bundle coincide with the orientation of  $S^7$ . Therefore, a 3-sphere  $F_i$  which bounds the fibre of a tubular neighborhood of  $S_i$  in  $S^7$  inherits an orientation and thus provides a generator  $e_i$  for  $H_3(S^7 \setminus S_i, \mathbf{Z}) \cong \mathbf{Z}$ ,  $i = 1, \dots, b'$ . For  $i \neq j$ , the image of the fundamental class  $[S_i]$  in  $H_3(S^7 \setminus S_j, \mathbf{Z})$  is of the form  $\lambda_{ij} \cdot e_j$ . The integer  $\lambda_{ij}$  is called *the linking number of  $S_i$  and  $S_j$* .

For  $i = 1, \dots, b'$ , the manifold  $S^7 \setminus \bigcup_{j \neq i} S_j$  is up to dimension 5 homotopy equivalent to  $\bigvee_{j \neq i} F_j$ , and

$$\pi_3(S^7 \setminus \bigcup_{j \neq i} S_j) \cong \pi_3(\bigvee_{j \neq i} F_j) \cong \bigoplus_{j \neq i} H_3(S^7 \setminus S_j, \mathbf{Z}).$$

Under this identification, we have  $[g_i] = \sum_{j \neq i} \lambda_{ij} \cdot e_j$ . The  $[g_i]$  determine the ambient isotopy class of the given link (3.9), and we deduce

**PROPOSITION 4.1.** *The linking numbers  $\lambda_{ij}$ ,  $1 \leq i < j \leq b'$ , determine the given link up to ambient isotopy.*

The sphere  $S_i$  bounds a 4-dimensional disc  $D_i^-$  in  $D^8$ ,  $i = 1, \dots, b'$ , which we equip with the induced orientation. We may, furthermore, assume

that the  $D_i^-$  intersect transversely in the interior of  $D^8$ . Then the  $\lambda_{ij}$  coincide with the intersection numbers  $D_i^- \cdot D_j^-$ ,  $1 \leq i < j \leq b'$ . For an intuitive proof (in dimension 4), see [28], p.67. Now, every disc  $D_i^-$  is completed by the core disc  $D_i^+$  of the  $i^{\text{th}}$  4-handle to an embedded 4-sphere  $\Sigma_i$  in  $T$ ,  $i = 1, \dots, b'$ , and, since all the core discs are pairwise disjoint, the  $\lambda_{ij}$  coincide with the intersection numbers  $\Sigma_i \cdot \Sigma_j$ ,  $1 \leq i < j \leq b'$ . Finally,  $X$  is obtained by gluing an 8-disc to  $T$  along  $\partial T$ , and the spheres  $\Sigma_i$  represent the elements of the chosen basis  $\underline{b}$  of  $H_4(X, \mathbf{Z})$ . Identifying the intersection ring with the cohomology ring of  $X$  via Poincaré-duality, we see

COROLLARY 4.2. *The linking numbers  $\lambda_{ij}$  coincide with the cup products  $(y_i \cup y_j)[X]$ ,  $1 \leq i < j \leq b'$ , i.e., the link of the attaching spheres is determined up to ambient isotopy by the basis  $\underline{b}$  and the cup products.*

As we have remarked before, the normal bundles of the  $S_i$  are trivial, whence there exist embeddings  $f_i^0: S^3 \times D^4 \rightarrow S^7$  with  $f_i^0|_{S^3 \times \{0\}} = g_i$ ,  $i = 1, \dots, b'$ . From the uniqueness of tubular (in differential topology) or regular (in piecewise linear topology) neighbourhoods, every other embedding  $f_i: S^3 \times D^4 \rightarrow S^7$  with  $f_i|_{S^3 \times \{0\}} = g_i$  is ambient isotopic to one of the form  $f_i^{[h_i]} := ((x, y) \mapsto (x, h_i \cdot y))$ ,  $[h_i] \in \pi_3(\text{SO}(4))$ ,  $i = 1, \dots, b'$ . Corollary 3.14 implies that we can choose the  $f_i^0$ ,  $i = 1, \dots, b'$ , in such a way that the following holds:

LEMMA 4.3. *Suppose  $T$  is obtained by attaching 4-handles along  $f_i^{[h_i]}$  with  $[h_i] = k_1^i \alpha_3 + k_2^i \beta_3$ ,  $i = 1, \dots, b'$ , then*

$$\Sigma_i \cdot \Sigma_i = k_2^i \quad \text{and} \quad p_1(T_{T|\Sigma_i}) = \pm(2k_2^i + 4k_1^i).$$

This shows that also the framed link used for constructing  $T$  and  $X$  is determined by the system of invariants associated to  $(X, \underline{y})$ , proving the injectivity in Part i) of the theorem. Moreover, the assertion about the fibres in Part ii) is clear.

Conversely, given a system  $Z$  of invariants in  $Z(0, b')$ , satisfying relation (2), there exists a based 3-connected manifold  $(X, \underline{y})$  realizing  $Z$ . Indeed, by the above identification of the invariants,  $Z$  determines a framed link in  $S^7$  and thus the manifold  $T := D^8 \cup H_1^4 \cup \dots \cup H_{b'}^4$ . The boundary of  $T$  is a 7-dimensional homotopy sphere ([17], (12.2), p.119) and, therefore, piecewise linearly homeomorphic to  $S^7$ . Hence,  $X = T \cup_{S^7} D^8$  is a piecewise linear manifold with the desired system of invariants, settling Part i). If, in

addition, relation (3) holds, then [18] ensures that  $X$  will carry a smooth structure (compare Theorem A.4 of [24]), finishing the proof of Part ii).  $\square$

#### 4.2 THE DETERMINATION OF $W_4$ IN THE GENERAL CASE

We have a handle decomposition  $W_0 \subset W_2 \subset W_4 \subset W_6 \subset X$  of  $X$  providing preferred bases  $\underline{b}$  of  $H_2(X, \mathbf{Z})$  and  $\underline{c}$  of  $H_4(X, \mathbf{Z})$ , respectively. Let  $\underline{x}$  and  $\underline{y}$  be the dual bases of  $H^2(X, \mathbf{Z})$  and  $H^4(X, \mathbf{Z})$ , respectively. Finally, let  $\underline{y}^*$  be the basis of  $H^4(X, \mathbf{Z})$  which is dual to  $\underline{y}$  via  $\gamma_X$ .

We find  $\partial W_2 \cong \#_{i=1}^b (S^2 \times S^5)$ , and  $W_4$  is determined by the ambient isotopy class of a framed link of 3-spheres in  $\partial W_2$  with  $b'$  components. Let  $f_k: S^3 \times D^4 \rightarrow \partial W_2$  be the  $k^{\text{th}}$  component of that link and  $g_k := f_k|_{S^3 \times \{0\}}$ ,  $k = 1, \dots, b'$ . In the notation of Section 3.6, we write  $[g_k] \in \pi_3(\partial W_2 \setminus \bigcup_{k \neq j} S_j)$  in the form  $(l_i^k, i = 1, \dots, b, l_{ij}^k, 1 \leq i < j \leq b; \lambda_{kj}, j \neq k)$ ,  $k = 1, \dots, b'$ . To see the significance of the  $l_i^k$  and  $l_{ij}^k$ , note that, by Remark 3.4,  $W_2 \cup H_k^4 \subset X$  is homotopy equivalent to  $(\bigvee_{i=1}^b S^2) \cup_{g_k} D^4$ . The cohomology ring of that complex has been computed in Proposition 3.11, so that the naturality of the cup product implies the following formulae for the cup products in  $X$ :

$$x_i \cup x_j = \sum_{k=1}^{b'} l_{ij}^k \cdot y_k^*, \quad i \neq j,$$

$$x_i \cup x_i = \sum_{k=1}^{b'} l_i^k \cdot y_k^*, \quad i = 1, \dots, b.$$

Therefore, the  $l_i^k$  and  $l_{ij}^k$  are determined by  $\delta_X$  and  $\gamma_X$  (used to compute  $\underline{y}^*$ ), in fact  $l_i^k = \gamma_X(\delta(x_i \otimes x_i) \otimes y_k)$  and  $l_{ij}^k = \gamma_X(\delta(x_i \otimes x_j) \otimes y_k)$ .

To determine the  $\lambda_{ij}$  and the framings, we proceed as follows: Look at the embedding  $\#_{i=1}^b (S^2 \times S^5) \hookrightarrow X$ . There exist  $b$  embedded 2-spheres  $S_1^2, \dots, S_b^2$  which represent the basis  $\underline{b}$  and which do not meet the given link. Finally,  $\#_{i=1}^b (S^2 \times S^5)$  obviously possesses a regular neighborhood in  $X$  which is homeomorphic to  $\#_{i=1}^b (S^2 \times S^5) \times D^1$ . Thus, we can perform "surgery in pairs" as described in Section 3.1. The result is a 3-connected manifold  $X^*$  containing  $S^7$ . It is by construction the manifold obtained from the framed link in  $S^7$  derived from the given one in  $\#_{i=1}^b (S^2 \times S^5)$  (cf. Section 4.1). We will be finished, once we are able to compare the invariants of  $X$  to those of  $X^*$ . To do so, we look at the *trace of the surgery*, i.e., at  $Y = (X \times I) \cup H_1^5 \cup \dots \cup H_{b'}^5$ , the 5-handles being attached along tubular neighborhoods of the  $S_i \times \{1\}$  in  $X \times \{1\}$ . Then  $\partial Y = X \sqcup \bar{X}^*$ .