4.3 Manifolds with given invariants

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The Mayer-Vietoris sequence provides the isomorphisms

$$H_4(X,\mathbf{Z}) \cong H_4\big(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\big) \cong H_4(X^*, \mathbf{Z}).$$

Set $H := H_4\big(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\big)$. By Lefschetz duality ([5], (28.18)), there is for each $q \in \mathbf{N}$ a diagram (omitting $\mathbf{Z}$-coefficients)

$$(4)\qquad \begin{array}{ccccccc} H^{q-1}(Y) & \longrightarrow & H^{q-1}(\partial Y) & \longrightarrow & H^q(Y, \partial Y) & \longrightarrow & H^q(Y) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ H_{10-q}(Y, \partial Y) & \longrightarrow & H_{9-q}(\partial Y) & \longrightarrow & H_{9-q}(Y) & \longrightarrow & H_{9-q}(Y, \partial Y) \end{array}$$

where the left square commutes up to the sign $(-1)^{q-1}$ and the other two commute. We first use it in the case $q = 5$. Look at the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\cong} & H_4(X^*, \mathbf{Z}) \\ \downarrow\cong & & \downarrow \\ H_4(X, \mathbf{Z}) & \longrightarrow & H_4(Y, \mathbf{Z}), \end{array}$$

in which all arrows are injective, because $H_5(Y, X; \mathbf{Z}) = 0 = H_5(Y, X^*; \mathbf{Z})$ (cf. [17], p. 198). Using the identification $H_4(\partial Y, \mathbf{Z}) = H \oplus H$, we find

$$(5)\qquad \mathrm{Im}\big(H_5(Y, \partial Y; \mathbf{Z})\big) = \{\, (y, -y) \in H \oplus H \,\}.$$

Similar considerations apply to the case $q = 9$. Taking into account that X^* sits in Y with the reversed orientation, (4) shows that the forms γ_X and γ_{X^*}, both defined with respect to the preferred bases, coincide. In the same manner, the pullbacks of $p_1(Y)$ to $H^4(X, \mathbf{Z})$ and $H^4(X^*, \mathbf{Z})$, respectively, agree. Since X and X^* are the boundary components of Y, these pullbacks are $p_1(X)$ and $p_1(X^*)$, respectively, and we are done. □

4.3 Manifolds with given invariants

One might speculate, especially in view of the classification of E-manifolds in dimension 4 and 6, that the invariants δ_X, γ_X, and $p_1(X)$ might suffice to classify E-manifolds with $w_2(X) = 0$ in dimension 8. However, Lemma 3.6 shows that these invariants determine only W_4 and we still have the choice of an isomorphism in gluing $\#_{i=1}^{b}(S^2 \times S^5)$ to W_4, and different gluings may lead to different results. The following example, which was communicated to me by J.-C. Hausmann, illustrates this phenomenon.

EXAMPLE 4.4. One has $\pi_5(\mathrm{SO}(3)) \cong \mathbf{Z}_2$ [32]. Therefore, there are two different S^2-bundles over S^6, call them $X := S^6 \times S^2$ and $X' := S^6 \mathbin{\tilde{\times}} S^2$. Obviously, X and X' are spin-manifolds with trivial invariants, but one computes $\pi_5(X) \cong \mathbf{Z}_2$ and $\pi_5(X') = \{0\}$.

Fix b, b', and a system Z of invariants in the image of the map $Z^{\mathrm{PL}(C^\infty)}(b, b')$. As we have seen, Z determines a certain manifold W_4 whose boundary is diffeomorphic to $\#_{i=1}^{b}(S^2 \times S^5)$ together with a basis $\underline{b}$ for $H_2(\partial W_4, \mathbf{Z})$. Let $\underline{b}_0$ be the natural basis for $H_2(\#_{i=1}^{b}(S^2 \times S^5), \mathbf{Z})$, and denote by $\mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$ the set of piecewise linear (smooth) isomorphisms $f\colon \#_{i=1}^{b}(S^2 \times S^5) \longrightarrow \partial W_4$ with $f_*(\underline{b}_0) = \underline{b}$. Our results show that every based piecewise linear (smooth) manifold $(X, \underline{x}, \underline{y})$ with system of invariants Z is piecewise linearly (smoothly) isomorphic to a manifold of the form

$$X(f) := \partial W_4 \cup_f \#_{i=1}^{b}(S^2 \times S^5) \quad \text{for some } f \in \mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$$

with its given bases for $H^2(X(f), \mathbf{Z})$ and $H^4(X(f), \mathbf{Z})$. Conversely, every manifold of the form $X(f)$ is a piecewise linear (smooth) based E-manifold with invariants Z.

Now, suppose we are given $f, f' \in \mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$, such that $X(f)$ and $X(f')$ are isomorphic as piecewise linear (smooth) based manifolds. We claim that we can find an isomorphism $\varphi\colon X(f) \longrightarrow X(f')$ with $\varphi(W_4) = W_4$. For this, look at the handle decomposition $W_0 \subset W_2 \subset W_4$. Since W_0 is just an embedded 8-disc in $X(f)$ and $X(f')$, respectively, we can choose φ with $\varphi(W_0) = W_0$. Let $l \subset \partial W_0$ be the framed link for attaching the 2-handles. Then $\varphi(l)$ and l are isotopic. Therefore, we can find a level preserving diffeomorphism $\widetilde{\psi}\colon [-1,1] \times \partial W_0 \longrightarrow [-1,1] \times \partial W_0$ with $\widetilde{\psi}|_{\{\pm 1\} \times \partial W_0} = \mathrm{id}_{\partial W_0}$ and $\widetilde{\psi}|_{\{0\} \times \partial W_0}(\varphi(l)) = l$. If we choose a tubular neighborhood ($\cong [-1,1] \times \partial W_0$) of ∂W_0 in $X(f')$, we can use $\widetilde{\psi}$ to define an automorphism $\psi\colon X(f') \longrightarrow X(f')$ with $\psi(\varphi(l)) = l$. Thus, $\psi \circ \varphi$ maps W_2 onto W_2. A similar argument shows that we can achieve $\varphi(W_4) = W_4$.

Let $\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(\#_{i=1}^b(S^2\times D^6))$ be the group of piecewise linear (smooth) automorphisms g of $\#_{i=1}^b(S^2\times D^6)$ with $H^2(g,\mathbf{Z})=\mathrm{id}$ and similarly define $\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(W_4)$. Then we have just established

PROPOSITION 4.5. *The set of isomorphy classes of based piecewise linear (smooth) E-manifolds with invariants Z is in bijection to the set of equivalence classes in* $\mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$ *with respect to the equivalence relation coming from the group action*

$$\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(W_4)\times\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(\#_{i=1}^b(S^2\times D^6))\times\mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}\longrightarrow\mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$$
$$(h,g,f)\longmapsto h|_{\partial W_4}\circ f\circ g|^{-1}_{\#_{i=1}^b(S^2\times S^5)}\,.$$

We shall see in Lemma 5.1 that $\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2\times D^6))$ contains the commutator subgroup of $\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2\times S^5))$.

COROLLARY 4.6. *The set of isomorphy classes of based piecewise linear E-manifolds with* $b_2=b$ *and* $b_4=0$ *is in bijection to the abelian group*

$$\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2\times S^5))/\,\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2\times D^6))\,.$$

I have been informed by experts that the structure of the groups $\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(\#_{i=1}^b(S^2\times S^5))$ and $\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(\#_{i=1}^b(S^2\times D^6))$ has not yet been determined and that this would be a rather difficult task. Therefore, we choose the viewpoint of framed links in order to finish our considerations. In Theorem 5.2, we will then use this viewpoint to compute the group $\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2\times S^5))/\,\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2\times D^6))$.

As above, let $(X,\underline{x},\underline{y})$ be a based piecewise linear (smooth) E-manifold with zero second Stiefel-Whitney class and system of invariants $Z_{(X,\underline{x},\underline{y})}=(\delta,\gamma,p)$. We have seen that we can find a framed link l_X of 2-spheres in X which represents the basis $\underline{x}$ and perform surgery along this link in order to get a 3-connected piecewise linear (smooth) based manifold $(X^*,\underline{y})$ together with a framed link $l^*_{X'}$ of 5-spheres in it. If $(X',\underline{x}',\underline{y}',l_{X'})$ is another such object where $(X',\underline{x}',\underline{y}')$ is isomorphic to $(X,\underline{x},\underline{y})$, then clearly we can find an isomorphism $\varphi\colon(X,\underline{x},\underline{y})\longrightarrow(X',\underline{x}',\underline{y}')$ with $\varphi(l_X)=l_{X'}$. Such an isomorphism φ yields, after surgery, an isomorphism $\varphi^*:(X^*,\underline{y})\longrightarrow(X'^*,\underline{y}')$ with $\varphi^*(l^*_{X'})=l^*_{X'^*}$. In particular, the manifold $(X^*,\underline{y})$ is determined up to piecewise linear (smooth) isomorphy. We call it the *type of* $(X,\underline{x},\underline{y})$. Note that this notion matters only in the smooth case, by Theorem 2.2.

To summarize, we have

PROPOSITION 4.7. *The set of isomorphy classes of based piecewise linear (smooth) E-manifolds of type* $(X^*, \underline{y})$ *is in bijection to the set of equivalence classes of framed links of* 5*-spheres in* X^* *where two such links* l *and* l' *are considered* equivalent, *if there is a piecewise linear (smooth) automorphism* $\varphi^* \colon (X^*, \underline{y}) \longrightarrow (X^*, \underline{y})$ *with* $\varphi^*(l) = l'$.

EXAMPLE 4.8. The group $\mathbf{Z}_2^{\oplus b}$ acts freely on the set of isotopy classes of framed links of b spheres of dimension 5 in X^* by altering the framings of the components. Note that the two possible framings of the trivial bundle on a 5-sphere are distinguished by the fact that one extends over D^6 and the other does not. This property is preserved under piecewise linear homeomorphisms, so that we conclude that $\mathbf{Z}_2^{\oplus b}$ acts also freely on the set of equivalence classes of framed links of b spheres of dimension 5 in X^*.

Note that this completes the classification of Spin-E-manifolds of dimension eight with second Betti number one.

Let us look at manifolds of type S^8. We claim that two framed links l and l' of 5-spheres are equivalent in the above sense, if and only if they are isotopic. Clearly, after replacing l and l' by isotopic links, we may assume that both of them are contained in the Southern hemisphere and that φ^* is the identity on the Northern hemisphere. Now, choose a representative $\varphi^\dagger$ for the isotopy class of ${\varphi^*}^{-1}$ which is the identity on the Southern hemisphere. Then $\varphi^\dagger \circ \varphi^*$ is isotopic to the identity and carries l into l'.

For differentiable manifolds, the operation $X \longmapsto X\#\Sigma$, Σ an exotic 8-sphere, establishes a bijection between the set of isomorphy classes of based smooth E-manifolds of type S^8 and the set of isomorphy classes of based smooth E-manifolds of type Σ. We conclude

COROLLARY 4.9. i) *The set of isomorphy classes of based piecewise linear E-manifolds with* $b_2 = b$ *and* $b_4 = 0$ *is in bijection to the group* $\mathrm{FL}_b = \mathrm{L}_b \oplus \bigoplus_{i=1}^{b} \mathbf{Z}_2$.

ii) *The set of isomorphy classes of based smooth E-manifolds with* $b_2 = b$ *and* $b_4 = 0$ *is in bijection to the group* $\vartheta^8 \oplus \mathrm{FL}_b$.

Finally, we have to deal with those manifolds for which the cup form δ is trivial. Our investigations in Sections 3.6 and 4.2 show that the framed link of 3-spheres in ∂W_2 can be chosen to be contained in a small disc.

In other words, a manifold X with $\delta_X \equiv 0$ is piecewise linearly (smoothly) isomorphic $X^\dagger \# X^*$ where X^* is the type of X and $b_4(X^\dagger) = 0$. As our surgery arguments above reveal, an isomorphism between $X^\dagger \# X^*$ and $X'^\dagger \# X'^*$ can be chosen of the form $\varphi^\dagger \# \varphi^*$ where $\varphi^\dagger \colon X^\dagger \longrightarrow X'^\dagger$ and $\varphi^* \colon X^* \longrightarrow X'^*$ are isomorphisms. Therefore, the set of isomorphy classes of based piecewise linear E-manifolds of type X^* with $b_2 = b$ is in bijection to the set of isomorphy classes of based piecewise linear E-manifolds with $b_2 = b$ and $b_4 = 0$. The same goes for differentiable manifolds of type X^*, if X^* is not diffeomorphic to $X^* \# \Sigma$, Σ an exotic 8-sphere. Otherwise, we have to divide by the action of ϑ^8. This observation together with Corollary 4.9 settles Theorem 2.4. □

5. Structure of the group $\mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times S^5)\right) / \mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times D^6)\right)$

In this section we prove that $\mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times S^5)\right) / \mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times D^6)\right)$ is an abelian group which is, moreover, isomorphic to the group FL_b defined before. This result should be of some independent interest, especially because the group FL_b is quite well understood by Haefliger's work. For $b = 1$, we refer to [20] for more specific information.

We begin with the elementary

Lemma 5.1. *Let $k \in \mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times S^5)\right)$ be a commutator. Then k extends to an automorphism of $\#_{i=1}^b (S^2 \times D^6)$.*

Proof. For the proof, we depict $\#_{i=1}^b (S^2 \times S^5)$ as follows: Let V_i, $i = 1, \dots, b$, be b copies of $S^2 \times D^6$, and we join V_i and V_{i+1} by a tube $T_i \cong [-1,1] \times D^7$, $i = 1, \dots, b-1$. The result is a manifold W whose boundary is isomorphic to $\#_{i=1}^b (S^2 \times S^5)$. We make the following normalizations: Write ∂V_i as $(S^2 \times D^i_+) \cup (S^2 \times D^i_-)$, let n_i and s_i be the centers of D^i_+ and D^i_-, respectively, and set $S^i_+ := S^2 \times n_i$ and $S^i_- := S^2 \times s_i$, $i = 1, \dots, b$. Choose furthermore points $e_i \neq w_i$ in $(S^2 \times D^i_+) \cap (S^2 \times D^i_-)$, $i = 1, \dots, b$, and suppose that $\{-1\} \times D^7 \subset T_i$ is attached to a disc around w_i in ∂V_i and $\{1\} \times D^7 \subset T_i$ to a disc around e_{i+1} in ∂V_{i+1}, $i = 1, \dots, b-1$. Set $T := \bigsqcup_{i=1}^{b-1} T_i$.

Now, let $k = f \circ g \circ f^{-1} \circ g^{-1}$ with $f, g \in \mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times S^5)\right)$. As $H_2(h, \mathbf{Z})$ is the identity for every element $h \in \mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times S^5)\right)$ and $S^i_\pm$, $i = 1, \dots, b$, both represent the same basis for $H_2(\partial W, \mathbf{Z})$, h is isotopic to a map h' which satisfies either assumption (A) or (B) below.