# 4. Prime parts of torsion numbers

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## 4. Prime parts of torsion numbers

We recall Jensen's formula, a short argument for which can be found in [Yo86].

LEMMA 4.1 [Jensen's formula]. *For any complex number* $\alpha$,

$$\int_0^1 \log \left|\alpha - e^{2\pi i\theta}\right| d\theta = \log \max\{1, |\alpha|\}.$$

By Lemma 4.1 the Mahler measure $M(f)$ of a nonzero polynomial with complex coefficients can be computed as

$$\exp \int_0^1 \log \left|f(e^{2\pi i\theta})\right| d\theta .$$

This observation motivated the definition of Mahler measure for polynomials in several variables. (See [Bo81] or [EW99], for example.)

In [EF96], [Ev99] G.R. Everest and B. Ní Fhlathúin proved a $p$-adic analogue of Jensen's formula, which we describe. Assume that $\alpha$ is an algebraic integer lying in a finite extension $K$ of $\mathbf{Q}$. For every prime $p$ there is a $p$-adic absolute value $|\cdot|_p$, the usual Archimedean absolute value corresponding to $\infty$. We recall the definition (see [La65] for more details): If $p$ is a prime number, then $|p^r m/n|_p = 1/p^r$, where $r$ is an integer, and $m, n$ are nonzero integers that are not divisible by $p$. By convention, $|0|_p = 0$. Each $|\cdot|_p$ extends to an absolute value $|\cdot|_v$ on $K$. Let $\Omega_v$ denote the smallest field which is algebraically closed and complete with respect to $|\cdot|_v$. Let $\mathbf{T}_v$ denote the closure of the group of all roots of unity, which is in general locally compact. Note that if $p = \infty$, then $\Omega_v = \mathbf{C}$ and $\mathbf{T}_v = \mathbf{T}$. Everest and Fhlathúin define

$$M_{\mathbf{T}_v}(t-\alpha) = \exp \int_{\mathbf{T}_v} \log |t-\alpha|_v d\mu = \exp \lim_{r\to\infty} \frac{1}{r} \sum_{\zeta^r=1} \log |\zeta - \alpha|_v .$$

Here $\int$ denotes the Shnirelman integral, given by the limit of sums at the right, where one skips over the undefined summands. The above integral exists even if $\alpha \in \mathbf{T}_v$, in which case it can be shown to be zero. Moreover, one has

$$\int_{\mathbf{T}_v} \log |t-\alpha|_v d\mu = \log \max\{1, |\alpha|_v\}, \tag{4.1}$$

which Everest and Fhlathúin refer to as a $p$-adic analogue of Jensen's formula.

Recall that $b_r^{(p)}$ denotes the *p-component* of $b_r$, the largest power of $p$ that divides $b_r$. The *content* of $f \in \mathbf{Z}[t]$ is the greatest common divisor of the coefficients. Using (4.1) we will prove

THEOREM 4.2. *Let $(G,\chi)$ be an augmented group, and let $p$ be a prime.*

(i) *If $\mathcal{M}$ has a square matrix presentation and $\Delta(t) \neq 0$, then the sequence $\{b_{r_k}\}$ of pure torsion numbers satisfies*

$$\lim_{r_k \to \infty} \left(b_{r_k}^{(p)}\right)^{1/r_k} = (\text{content } \Delta)^{(p)}\,.$$

(ii) *If $\mathcal{M}$ is a direct sum of cyclic modules, then the sequence of all torsion numbers satisfies*

$$\lim_{r \to \infty} \left(b_{r}^{(p)}\right)^{1/r} = (\text{content } \Delta)^{(p)}\,.$$

(iii) *If $\mathcal{M}$ is torsion free as an abelian group, then*

$$\lim_{r \to \infty} \left(b_{r}^{(p)}\right)^{1/r} = 1\,.$$

EXAMPLE 4.3. For any positive integer $m$, consider the augmented group $(G,\chi)$ where $G$ is the Baumslag-Solitar group $\langle x, y \mid y^m x = x y^m \rangle$ and $\chi\colon G \to \mathbf{Z}$ maps $x \mapsto 1$ and $y \mapsto 0$. One verifies that $\mathcal{M} \cong \mathcal{R}_1/(m(t-1))$. The quotient module $\mathcal{M}_r$ is isomorphic to $\mathbf{Z}^r/A_r\mathbf{Z}^r$, where

$$A_r = \begin{pmatrix} m & 0 & 0 & 0 & \cdots & -m \\ -m & m & 0 & \cdots & & 0 \\ 0 & -m & m & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & & \cdots & -m & m \end{pmatrix}.$$

The matrix is equivalent by elementary row and column operations to the diagonal matrix

$$\begin{pmatrix} m & & & \\ & \ddots & & \\ & & m & \\ & & & 0 \end{pmatrix}.$$

Hence $\mathcal{M}_r \cong \mathbf{Z} \oplus (\mathbf{Z}/m)^{r-1}$, and so $b_r = m^{r-1}$ for all $r$. Consequently,

$$\lim_{r \to \infty} (b_r^{(p)})^{1/r} = m^{(p)}\,.$$

The Alexander polynomial of any knot is nonzero, and its coefficients are relatively prime. Hence the following corollary is immediate from Theorem 4.2 (iii).

COROLLARY 4.4. *For any knot $k$ and prime $p$,*

$$\lim_{r\to\infty}(b_r^{(p)})^{1/r}=1\,.$$

Theorem 2.10 and Corollary 4.4 imply that whenever the Alexander polynomial of $k$ has Mahler measure greater than 1, infinitely many distinct primes occur in the factorization of the torsion numbers $b_r$. In other words, the homology groups $H_1(M_r,\mathbf{Z})$ display nontrivial $p$-torsion for infinitely many primes $p$. Since the sequence $\{b_r\}$ is a division sequence, the number of prime factors of $b_r$ is unbounded.

What about the case in which the Alexander polynomial of $k$ has Mahler measure equal to 1 ? The argument of Section 5.7 of [Go72] shows that the number of prime factors remains unbounded as long as the Alexander polynomial does not divide $t^M-1$ for any $M$. If it does divide, then the torsion numbers $b_r$ are periodic by Section 5.3 of [Go72] (see also Corollary 2.2 of [SiWi00]). Hence we obtain

COROLLARY 4.5. *For any knot, either the torsion numbers $b_r$ are periodic or else for any $N>0$ there exists an $r$ such that the factorization of $b_r$ has at least $N$ distinct primes.*

The proof of Theorem 4.2 requires the following lemma.

LEMMA 4.6. *If $f(t)=c_0t^n+\cdots+c_{n-1}t+c_n$ is a nonzero polynomial in $\mathbf{Z}[t]$ with roots $\lambda_1\cdots,\lambda_n$ (not necessarily distinct) in $\Omega_v$, then*

$$|c_0|_v\prod_{i=1}^{n}\max\{1,|\lambda_i|_v\}=|\mathrm{content}\,f|_v\,.$$

*Proof.* The argument that we present is found in [LW88]. Set $a_j=c_j/c_0$ for $0\le j\le n$, so $f(t)=c_0(t^n+a_1t^{n-1}+\cdots+a_n)$. Each $a_j$ is an elementary symmetric function of the roots $\lambda_i$, namely the sum of products of roots taken $j$ at a time. Using the ultrametric property

$$|x+y|_v=\max\{|x|_v,|y|_v\}\,,$$

we see that if exactly $k$ values of $|\lambda_i|_v$ are greater than 1, then

$$\max_j|a_j|_v=|a_k|_v=\prod_{j=1}^{n}\max\{1,|\lambda_j|_v\}\,.$$

But

$$\max_j |a_j|_v = \max\{1, \left|\frac{c_1}{c_0}\right|_v, \ldots, \left|\frac{c_n}{c_0}\right|_v\} = \frac{|\text{content } f|_v}{|c_0|_v}.$$

Hence the lemma is proved. □

*Proof of Theorem 4.2.* In case (i), the pure torsion number $b_{r_k}$ is equal to $\left|\prod_{\zeta^{r_k}=1} \Delta(\zeta)\right|$. We have

$$|b_{r_k}|_v = \left|\prod_{\zeta^{r_k}=1} \Delta(\zeta)\right|_v = |c_0|_v^{r_k} \prod_{\zeta^{r_k}=1} \prod_{j=1}^{n} |\zeta - \lambda_j|_v ,$$

where $c_0$ is the leading coefficient of $\Delta$ and $\lambda_1, \ldots \lambda_n$ are its roots. Hence

$$\begin{aligned} |b_{r_k}|_v^{1/r_k} &= |c_0|_v \prod_{\zeta^{r_k}=1} \prod_{j=1}^{n} |\zeta - \lambda_j|_v^{1/r_k} \\ &= |c_0|_v \prod_{j=1}^{n} \exp\Big(\frac{1}{r_k} \sum_{\zeta^{r_k}=1} \log|\zeta - \lambda_j|\Big) , \end{aligned}$$

so that

$$\lim_{r_k \to \infty} |b_{r_k}|_v^{1/r_k} = |c_0|_v \prod_{j=1}^{n} \exp \int_{\mathbf{T}_v} \log|t - \lambda_j|_v \, d\mu ,$$

which by equation (4.1) is equal to

$$|c_0|_v \prod_{j=1}^{n} \max\{1, |\lambda_j|_v\} .$$

By Lemma 4.6 this is equal to $|\text{content } \Delta|_v$. But for integers $n$ we have $n^{(p)} = |n|_v^{-1}$.

Now suppose $\mathcal{M}$ is cyclic. As in the proof of Theorem 3.8, we let $\gamma$ be the cyclotomic order of $\Delta$ and consider the subsequence of $b_r$ with $r$ in a fixed congruence class modulo $\gamma$. Then starting with Theorem 3.3 we may apply the argument above with $\Delta/\Phi$ in place of $\Delta$ to show that the limit of $(|b_r|^{(p)})^{1/r}$ along this subsequence is the $p$-component of the content of $\Delta/\Phi$. But content is multiplicative and cyclotomic polynomials have content 1, so the limit along all congruence classes is $(\text{content } \Delta)^{(p)}$. The result is immediate for direct sums of cyclic modules.

Finally, we can extend the result when $\mathcal{M}$ is torsion-free as an abelian group using Theorem 3.6. But for this case the content of $\Delta$ is 1. □