4. Homogeneity and holonomy

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

THEOREM 3.11 ([O4]). *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space with* $\operatorname{rank}(M) \geq 1$. *Then* M *is contained in a sphere.*

We summarize all the above facts in the following theorem.

THEOREM 3.12. *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space. Then,*

(i) $\operatorname{rank}(M) \geq 1$ *if and only if* M *is contained in a sphere;*

(ii) $\operatorname{rank}(M) \geq 2$ *if and only if* M *is an orbit of an* s*-representation.*

The next corollary uses the fact that the minimal homogeneous submanifolds of Euclidean spaces must be totally geodesic (see [D]).

COROLLARY 3.1. *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space with parallel mean curvature vector* H. *Then* $H \neq 0$ *and* M *is either minimal in a sphere or it is an orbit of an* s*-representation.*

4. HOMOGENEITY AND HOLONOMY

In this section we briefly relate homogeneity and holonomy. In particular, we are interested in the computation of the holonomy group in homogeneous situations. We shall put special emphasis on the tangent bundle of a homogeneous Riemannian manifolds and the normal bundle of a homogeneous submanifold of Euclidean space. But, in the first part, we will work in the framework of arbitrary homogeneous (pseudo)metric vector bundles with a connection. This is because, in our opinion, the main ideas are better understood in this context. Another reason is that one can prove, without extra efforts, very general results which could have applications to the pseudoriemannian case.

Let $E \xrightarrow{\pi} M$ be a finite dimensional real vector bundle over M with a covariant derivative operator ∇ (also called a connection), which corresponds, as usual, to a connection in the sense of distributions. More precisely, there exists a distribution $\mathcal{H}$ on TE such that

(1) $\mathcal{H} \oplus \mathcal{V} = TE$, where $\mathcal{V}$ is the vertical distribution;

(2) $(\mu_c)_*(\mathcal{H}_q) = \mathcal{H}_{\mu_c(q)}$, for all $c \in \mathbf{R}$, where μ_c is multiplication by c.

Let $\langle\ ,\ \rangle$ be a C^∞ metric on the fibres and let g be a Riemannian metric on M (in fact, $\langle\ ,\ \rangle$ and g need not be positive definite). We assume that there is a Lie group G which acts on E by bundle morphisms, whose induced action on M is by isometries and is transitive. Moreover, we assume that the action on E preserves both the metric on the fibres and the connection. A vector X in the Lie algebra $\mathcal{G}$ of G induces, in a natural way, a Killing vector field $\widetilde{X}$ both on E and M, i.e., if $\xi_p \in E$ (resp. $p \in M$) then $\widetilde{X}(\xi_p) := X\,.\,\xi_p := \frac{d}{dt}|_{t=0}\exp(tX)\xi_p$ (resp. $\widetilde{X}(p) := X\,.\,p := \frac{d}{dt}|_{t=0}\exp(tX)p$), where $\exp(tX)$ is the one parameter subgroup associated with X.

We will always keep in mind, as remarked above, the following two important cases:

(a) $M = G/H$ is a homogeneous Riemannian manifold, where G is a Lie subgroup of the isometry group $I(M)$, $E = TM$ is the tangent bundle and ∇ is the usual Levi-Civita connection.

(b) $M = G\cdot v$, where $v \in \mathbf{R}^n$ and G is a Lie subgroup of the isometry group $I(\mathbf{R}^n)$. Here, $E = \nu(M)$ is the normal bundle endowed with the usual normal connection $\nabla^\perp$.

The bundle E is endowed with the so-called Sasaki (Riemannian) metric $\tilde{g}$. Namely,

(i) $\mathcal{H}$ is perpendicular to the vertical distribution $\mathcal{V}$, defined by the tangent space to the fibres $E_q = \pi^{-1}(q)$.

(ii) The restriction of $\tilde{g}$ to $\mathcal{V}$ coincides with the metric on the fibres.

(iii) π is a Riemannian submersion.

The Sasaki metric may be regarded as follows. A curve $\tilde{c}(t)$ in E may be viewed as a section along the curve $c(t) = \pi(\tilde{c}(t))$. In this way, $\tilde{g}(\tilde{c}\,'(0), \tilde{c}\,'(0)) = \langle \frac{D}{dt}|_0\tilde{c}(t), \frac{D}{dt}|_0\tilde{c}\rangle + g(c'(0), c'(0))$.

Observe that G acts by isometries, with respect to the Sasaki metric, on E. As is well known, the fibres E_q, $q \in M$, are totally geodesic submanifolds of E. In fact, if $c(t)$ is a curve in M starting at q, then the parallel transport τ_t^c along $c(t)$ defines an isometry from E_q into $E_{c(t)}$. Let $\gamma(s)$ be a curve in E_q and consider $f(s,t) = \tau_t^c(\gamma(s))$. We have that $\langle \tau_t^c(\gamma'(s)), \tau_t^c(\gamma'(s))\rangle$ does not depend on t and so,

$$0 = \frac{\partial}{\partial t}\tilde{g}\,(\tfrac{\partial}{\partial s}f, \tfrac{\partial}{\partial s}f) = 2\tilde{g}\,(\tfrac{D}{\partial t}\tfrac{\partial}{\partial s}f, \tfrac{\partial}{\partial s}f) = 2\tilde{g}\,(\tfrac{D}{\partial s}\tfrac{\partial}{\partial t}f, \tfrac{\partial}{\partial s}f) = 2\langle A_{\frac{\partial}{\partial t}f}\tfrac{\partial}{\partial s}f, \tfrac{\partial}{\partial s}f\rangle\,,$$

where A denotes the shape operator of E_q as a submanifold of E. Then E_q is totally geodesic.

We now describe how the holonomy algebra (i.e., the Lie algebra of the holonomy group of the connection ∇ of the bundle $E \xrightarrow{\pi} M$) is linked with the group G. As we saw above the fibres E_q of the bundle E are totally geodesic. Then the projection on E_q of a Killing field $\widetilde{X}$ of E, induced by some $X \in \mathcal{G}$, gives a Killing field $B_q(X)$ of the fibre E_q. Observe that this projection vanishes at 0_q, so $B_q(X)$ belongs to $\mathfrak{so}(E_q)$, the Lie algebra of $\mathrm{SO}(E_q)$. The Lie algebra spanned by these $B_q(X)$ is included in the Lie algebra of the normalizer $N(\mathrm{Hol}_q)$ of the holonomy group Hol_q in $\mathrm{SO}(E_q)$. This is due to the following geometric reason: for any curve c in M and $g \in G$, $\tau_t^{g.c} = g.\tau_t^c.g^{-1}$, since G preserves the connection (and so $g.\mathrm{Hol}_p.g^{-1} = \mathrm{Hol}_{g.p}$, where Hol denotes the holonomy group of the connection on the bundle E).

Let τ_t^X be the flow on E associated to the horizontal component $[\widetilde{X}]^{\mathcal{H}}$ of the Killing field $\widetilde{X}$ (i.e. if $\xi_p \in E_p$, then $\tau_t^X(\xi_p)$ is the parallel transport of ξ_p along the curve $\exp(sX)\cdot p$ from 0 to t). Let F_t^X be the flow of the Killing field $\widetilde{X}$ on E, i.e., $F_t^X(\xi_p) := \exp(tX)\xi_p$. Then the fact that isometries and parallel transport are geometric objects implies that $\tau_t^X \circ F_s^X = F_s^X \circ \tau_t^X$. Taking into account this identity, one finds that $\phi_t := \tau_{-t}^X \circ F_t^X$ defines a one parameter group of isometries of E with the following properties: (i) $\phi_t(E_q) = E_q$, (ii) $\phi_t|_{E_q}$ belongs to $N(\mathrm{Hol}_q)$, the normalizer in $\mathrm{SO}(E_q)$ of the holonomy group Hol_q and (iii) $\phi_t|_{E_q} = e^{B_q(X)t}$, where $B_q(X)$ is the claimed projection of the Killing field X to E_q (i.e. $B_q(X)\xi_q = [X.\xi_q]^{\mathcal{V}}$, where $[\ ]^{\mathcal{V}}$ denotes vertical projection). Note that (iii) is a simple consequence of the general fact that if two flows F_t^X, F_t^Y commute then $F_t^X \circ F_t^Y = F_t^{X+Y}$.

The following theorem makes precise the above description and establishes, using the transitivity of G on M, the inclusion of the holonomy algebra into the Lie algebra generated by the $B_q(X)$.

THEOREM 4.1 ([OSv]). *The Lie algebra $\mathcal{L}_q$ generated by $\{B_q(X) : X \in \mathcal{G}\}$ contains the Lie algebra of the holonomy group* Hol_q *and is contained in the Lie algebra* $N(\mathrm{Hol}_q)$ *of its normalizer in* $\mathrm{SO}(E_q)$.

Proof. In order to illustrate better the main ideas we will only prove a simplified version of the theorem. The inclusion in the normalizer was observed before. Let L_q denote the Lie group associated to $\mathcal{L}_q$ and let $\xi_q \in E_q$. Let us consider $S_{\xi_q} := G \cdot L_q \cdot \xi_q \subset E$. Note that either $S_{\xi_p} \cap S_{\eta_q} = \varnothing$ or $S_{\xi_p} = S_{\eta_q}$, for all $\eta_p, \xi_q \in E$.

It is standard to show that S_{ξ_q} is a subbundle of E over M (of course not a vector subbundle). Observe that the connected component of the fibre at q

of S_{ξ_q} is $L_q \cdot \xi_q$, since the connected component of the isotropy subgroup G_q is contained in L_q. So, the restrictions $\widetilde{X}_{|S_{\xi_q}}$ and $[\widetilde{X}]^{\mathcal{V}}_{|S_{\xi_q}}$ are both tangent to S_{ξ_q} and hence the horizontal component $[\widetilde{X}]^{\mathcal{H}}_{|S_{\xi_q}}$ is also tangent to S_{ξ_q}. Since G acts transitively on M, $\{[\widetilde{X}]^{\mathcal{H}}(\xi_q) : X \in \mathcal{G}\}$ coincides with the horizontal space $\mathcal{H}_{\xi_q}$ (note that $\pi_*(\widetilde{X}) = X$).

Then $\mathcal{H}_\eta \subset T_\eta S_{\xi_q}$ for all $\eta \in S_{\xi_q}$. This implies that $\mathrm{Hol}^*_q \cdot \xi_q \subset L_q \cdot \xi_q$, where Hol^*_q is the connected component of Hol_q (i.e., the restricted holonomy group). In other words, any orbit of Hol^*_q is contained in an orbit of L_q. To get the inclusion $\mathrm{Hol}^* \subset L_q$ one has to carry out a similar argument but replacing E by the principal bundle over M of orthonormal basis of E. □

APPLICATIONS

- $E = TM$, the tangent bundle: in this case we will show that $B_q(X) = (\nabla \bar{X})_q$, where $\bar{X}(p) = X \cdot p$, $p \in M$ (cf. [N]). Indeed,

$$
\begin{aligned}
B_q(X) \cdot \xi &= \frac{D}{dt}_{|0} \exp(tX) \cdot \xi = \frac{D}{\partial t}_{|0} \frac{\partial}{\partial s}_{|0} \exp(tX) \cdot \gamma_\xi(s) \\
&= \frac{D}{\partial s}_{|0} \frac{\partial}{\partial t}_{|0} \exp(tX) \cdot \gamma_\xi(s) = \frac{D}{ds}_{|0} X \cdot \gamma_\xi(s) = \nabla_\xi \bar{X},
\end{aligned}
$$

where γ_ξ is the geodesic of M with initial condition $\xi \in E_p$.

If M is locally irreducible and the scalar curvature is not (identically) zero, then the restricted holonomy group Φ^*_q of M is non exceptional, i.e. it acts on T_qM as an s-representation (see [Sim, p. 229]). Then Φ^*_q coincides with the connected component of its normalizer in $\mathrm{SO}(T_qM)$. So, the Lie algebra of Φ^*_q is algebraically generated by $\{B_q(X) : X \in \mathcal{G}\}$. More generally, if M is not Ricci flat the same conclusion holds due to [K] and is now a consequence of next proposition. But Alekseevsky-Kimel′feld [AK] proved that a homogeneous Riemannian manifold cannot be Ricci flat, unless it is flat (a conceptual proof is due to Heintze and appeared in [BB, p. 553]). Then the holonomy algebra can always be calculated in this way for a locally irreducible M (the so-called Kostant's method). The following result is essentially due to Lichnerowicz and it is a consequence of Berger's list [B1]. Since it is difficult to find in the literature we include a simple proof.

PROPOSITION 4.1. *Let M be a Riemannian manifold which is irreducible at $q \in M$ and let $\mathfrak{g}$ be the Lie algebra of the local holonomy group Φ^{loc}_q at q. Let $\mathfrak{n}$ be the normalizer of $\mathfrak{g}$ in $\mathfrak{so}(T_qM)$. Then $\mathfrak{n}$ contains $\mathfrak{g}$ properly if and only if M is Kähler and Ricci flat near q.*

Proof. Let us endow $\mathfrak{so}(T_qM)$ with the usual scalar product $\langle A,B\rangle = -\operatorname{tr}(A.B)$. Assume that $\mathfrak{n} \neq \mathfrak{g}$. If we decompose orthogonally $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{k}$, then $\mathfrak{g}$ and $\mathfrak{k}$ are ideals of $\mathfrak{n}$ and so $[\mathfrak{g},\mathfrak{k}] = 0$. Now choose $0 \neq J_q \in \mathfrak{k}$. Then J_q^2 is a symmetric endomorphism which commutes with $\mathfrak{g}$. So, J_q^2 commutes with Φ_q^{loc} and then each eigenspace of J_q^2 defines a parallel distribution near q. Since M is locally irreducible at q we conclude, by de Rham's Decomposition Theorem, that $J_q^2 = -c^2 \,\mathrm{id}$. We may assume, by rescaling J_q, that $J_q^2 = -\,\mathrm{id}$. Extending J_q by parallelism we obtain a parallel almost complex structure J on M. Thus, M is Kähler near q. It is well-known [KN, Proposition 4.5, p. 149, vol. II] that the Ricci curvature Ric_M of a Kähler manifold M satisfies:

$$\operatorname{Ric}_M(X, JY) = \frac{\langle R_{X,Y}, J\rangle}{2}$$

If γ is any curve in a small neighbourhood of q joining q to p, and τ_γ is the parallel transport along γ, then

$$\langle R_{X_p,Y_p}, J_p\rangle = \langle \tau_\gamma^{-1} R_{X_p,Y_p}\tau_\gamma, J_q\rangle = 0$$

since $J_q \perp \mathfrak{g}$. So, M is Ricci flat near q.

The above two formulas, together with the Ambrose-Singer holonomy theorem also show the converse. □

• $E = \nu(M)$, the normal bundle of a submanifold of $\mathbf{R}^n$. Recall that in this case the non trivial part of the normal holonomy representation is an s-representation. Hence, the semisimple part of the normal holonomy group coincides with the connected component of its own normalizer (in the orthogonal group). If M is an irreducible submanifold which is not a curve, then the group G gives the parallel transport in $\nu_0(M)$ (the maximal parallel and flat subbundle of $\nu(M)$ (see [O3]). So, in this case, the Lie algebra of the normal holonomy group is algebraically generated by $\{B_q(X) : X \in \mathcal{G}\}$. Moreover, we have that $B_q(X)$ can be regarded as the projection to the affine subspace $q + \nu_q(M)$ of the Killing field of $\mathbf{R}^n$ (restricted to this normal space) induced by $X \in \mathcal{G}$. So, the normal holonomy group measures how far G is from acting polarly and how far M is from being a principal orbit (in which case this projection would be trivial from the definition of polarity).

Polar actions on the tangent bundle and symmetry

We recall briefly the general notions of polar and hyperpolar actions on Riemannian manifolds; for more details we refer to [Da, PT2, PT1, HPTT]. Let M be a complete Riemannian manifold and let G be a closed subgroup of the full group of isometries of M. A complete embedded and closed submanifold Σ of M is called a *section* if Σ does intersect any orbit of G in M and is perpendicular to orbits at intersection points. If there exists a section in M then the action of G is called a *polar action.* Observe that from a section we can obtain, by means of the group, sections which contain any given point. An action is called *hyperpolar* if it is polar and a section is in addition flat. Of course in the case of $\mathbf{R}^n$ these two concepts coincide.

Let now M be a complete simply connected Riemannian manifold and let TM be its tangent bundle endowed with the Sasaki metric. We will regard M as the (Riemannian) embedded submanifold of TM which consists of the zero vectors. We have the following characterization of symmetric spaces in terms of polar (or equivalently, hyperpolar) actions on TM. The following result was obtained by J. Eschenburg and the third author when writing the article [EO].

THEOREM 4.2. *Let M be a simply connected complete Riemannian manifold. Then the tangent bundle TM admits a polar action having M as an orbit if and only if M is symmetric.*

Proof. Assume M is irreducible. Let G act polarly on TM and $G{\cdot}0_q = M$. If Σ is a section for this action with $q \in \Sigma$ then $\Sigma \subset T_qM$, since horizontal and vertical distributions are perpendicular with respect to the Sasaki metric. Since Σ meets G-orbits perpendicularly, we have that the horizontal distribution of TM is tangent to the G-orbits. Then the parallel transport of any $v \in T_qM$ belongs to $G \cdot v$. If the codimension of $G \cdot v$ is greater than 1, then the holonomy group does not act transitively on the (unit) sphere of T_qM. Hence M is symmetric by the theorem of Berger [B1, Sim]. If $G \cdot v$ has codimension 1 then M must be two point homogeneous and hence symmetric by [Wa] (for a conceptual proof see [Sz]). If $M = M_1 \times \cdots \times M_k$ is reducible, by projecting Killing vector fields to the factors we obtain a bigger group, let us say $\widetilde{G} = G_1 \times \cdots \times G_k$ and such that G_i acts polarly on M_i.

Let us show the converse. As we noted in Section 2, the transvection group Tr(N) acts transitively on any holonomy bundle. Then the polarity follows from the fact the holonomy representation acts polarly. □

It follows from the above results that an irreducible homogeneous space in which holonomy agrees with isotropy must be symmetric.