5. Torsion numbers and links

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5. Torsion numbers and links

A *link* is a finite collection $l = l_1 \cup \cdots \cup l_\mu$ of pairwise disjoint knots in the 3-sphere. If a direction is chosen for each component l_i, then the link is *oriented.* Equivalence for links, possibly oriented, is defined just as for knots.

The abelianization of the group $G = \pi_1(S^3 - l)$ is free abelian of rank μ with generators $t_1, \ldots, t_\mu$ corresponding to oriented loops having linking number one with corresponding components of l. When $\mu > 1$ there are infinitely many possible epimorphisms from G to the integers.

When l is oriented there is a natural choice for χ, sending each generator t_i to $1 \in \mathbf{Z}$. In this way we associate to l an augmented group (G, χ). As in the special case of a knot, $\mathcal{M}$ has a square presentation matrix, and it is isomorphic to the first homology group of the infinite cyclic cover of $S^3 - l$ corresponding to χ. Again as in the case of a knot, there is a sequence of r-fold cyclic covers M_r of S^3 branched over l. However, $H_1(M_r; \mathbf{Z})$ is isomorphic to $\mathcal{M}/(t^{r-1} + \cdots + t + 1)\mathcal{M}$ rather than $\mathcal{M}/(t^r - 1)\mathcal{M}$ (see [Sa79]). In the case of a knot the two modules are well known to be isomorphic (see Remark 5.4(i)).

Motivated by these observations we make the following definitions. Let $\widetilde{\mathcal{M}}_r$ denote the quotient module $\mathcal{M}/\nu_r\mathcal{M}$, where $\nu_r = t^{r-1} + \cdots + t + 1$.

Definition 5.1. Let (G, χ) be an augmented group. The r^{th} *reduced torsion number* $\widetilde{b}_r$ is the order of the torsion submodule $T\widetilde{\mathcal{M}}_r$. The r^{th} *reduced Betti number* $\widetilde{\beta}_r$ is the rank of $\widetilde{\mathcal{M}}$.

As before, we may also speak of the reduced torsion and Betti numbers of a finitely generated $\mathcal{R}_1$-module $\mathcal{M}$.

Many results of Section 2 apply to reduced torsion and Betti numbers with only slight modification. For example, an argument similar to the proof of Proposition 2.1 shows that $\widetilde{\beta}_r$ is the number of zeros of the Alexander polynomial which are roots of unity and different from 1, each zero counted as many times as it occurs in the elementary divisors Δ_i/Δ_{i+1}; hence $\widetilde{\beta}_r$ is periodic in r. Also, when $\widetilde{\beta}_r = 0$ the reduced torsion number $\widetilde{b}_r$ is equal to the absolute value of the resultant of Δ and ν_r.

Lemma 5.2. *Assume that* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is an exact sequence of finitely generated abelian groups. If* A *is finite, then the induced sequence*

$$0 \to A \xrightarrow{f} TB \xrightarrow{g} TC \to 0$$

is also exact.

Proof. The only thing to check is surjectivity of g. Since the alternating sum of the ranks of A, B and C is zero and A is finite, the ranks of B and C are equal. By Lemma 2.3 the homomorphism g maps TB onto TC. □

PROPOSITION 5.3. *Assume that the finitely generated $\mathcal{R}_1$-module $\mathcal{M}$ has a square presentation matrix. If $\Delta(1) \neq 0$, then for every r,*

$$\widetilde{\beta}_r = \beta_r, \qquad \widetilde{b}_r = \frac{b_r}{\delta_r}, \tag{5.1}$$

where δ_r is a divisor of $|\Delta(1)|$. Moreover, $\delta_{r+\gamma} = \delta_r$, for all r, where γ is the cyclotomic order of Δ.

Proof. Consider the sequence

$$\mathcal{M}_1 \xrightarrow{\nu_r} \mathcal{M}_r \xrightarrow{\pi} \widetilde{\mathcal{M}}_r \to 0,$$

where ν_r is multiplication by $\nu_r = t^{r-1} + \cdots + t + 1$, and π is the natural projection. It is easy to see that the sequence is exact. From it we obtain the short exact sequence

$$0 \to \mathcal{M}_1/\ker \nu_r \xrightarrow{\nu_r} \mathcal{M}_r \xrightarrow{\pi} \widetilde{\mathcal{M}}_r \to 0.$$

Here ν_r also denotes the induced quotient homomorphism. Since $\Delta(1) \neq 0$, the module $\mathcal{M}_1$ is finite and hence $\beta_r = \widetilde{\beta}_r$. The order of $\mathcal{M}_1$ is $|\Delta(1)|$, and hence the order of $\mathcal{M}_1/\ker \nu_r$ is a divisor δ_r. The second statement of (5.1) follows from Lemmas 5.2 and 3.7.

It remains to show that δ_r has period γ. For this let $0 \neq a \in \mathcal{M}$. The coset $\bar{a} \in \mathcal{M}_1$ is in the kernel of ν_r if and only if there exists $b \in \mathcal{M}$ such that $\nu_r(a-(t-1)b) = 0$. Clearly this is true if and only if $\nu_{(\gamma,r)}(a-(t-1)b) = 0$, where (γ, r) denotes the gcd of γ and r. Hence the kernel of ν_r is equal to the kernel of $\nu_{(\gamma,r)}$, and the periodicity of δ_r follows. □

REMARKS 5.4.

(i) If G is a knot group, then any two meridianal generators are conjugate. Consequently $\mathcal{M}_1$ is trivial. Proposition 5.3 implies that in this case, the torsion numbers b_r and $\widetilde{b}_r$ are equal for every r.

(ii) It is well known that for any oriented link $l = l_1 \cup l_2$ of two components, $|\Delta(1)|$ is equal to the absolute value of the linking number $\mathrm{Lk}(l_1, l_2)$. (See Theorem 7.3.16 of [Ka96].)

PROPOSITION 5.5. *Let $\mathcal{M}$ be a finitely generated $\mathcal{R}_1$-module with a square presentation matrix. Assume that $\Delta(t) = (t-1)^q g(t)$, with $g(1) \neq 0$. If p is a prime that does not divide $g(1)$, then*

$$\widetilde{\beta}_{p^k} = 0, \quad \widetilde{b}^{(p)}_{p^k} = p^{qk},$$

for every $k \geq 1$.

The proof of Proposition 5.5 requires:

LEMMA 5.6. *Let $g(t)$ be a polynomial with integer coefficients, and assume that p is a prime. If p does not divide $g(1)$, then p does not divide* $\mathrm{Res}(g, t^{p^k} - 1)$ *for any positive integer k.*

Proof of Lemma 5.6. Assume that p does not divide $g(1)$. Recall that $\Phi_n(t)$ denotes the n^{th} cyclotomic polynomial. From the formula

$$\prod_{\substack{d|n \\ d>1}} \Phi_d(1) = \nu_n(1) = n,$$

we easily derive

$$\Phi_d(1) = \begin{cases} 0 & \text{if } d = 1 \\ q & \text{if } d = q^k > 1,\ q \text{ prime} \\ 1 & \text{other } d. \end{cases}$$

Consequently, Φ_{p^k} does not divide g for any $k > 0$, and so $\mathrm{Res}(g, t^{p^k} - 1) \neq 0$. The module $\mathcal{H} = \mathcal{R}_1/(g, t^{p^k} - 1)$ has order $\left|\mathrm{Res}(g, t^{p^k} - 1)\right|$, and it suffices to prove that $\mathcal{H} \otimes_{\mathbf{Z}} \mathbf{Z}/p$ is trivial. Now, $\mathcal{H} \otimes_{\mathbf{Z}} \mathbf{Z}/p$ is isomorphic to the quotient of the PID $(\mathbf{Z}/p)[t, t^{-1}]$ by the ideal generated by the greatest common divisor of g and $t^{p^k} - 1$ in this ring. But $t^{p^k} - 1 = (t-1)^{p^k}$ in this ring, and $t - 1$ does not divide g since p does not divide $g(1)$. So the gcd is 1, and $\mathcal{H} \otimes_{\mathbf{Z}} \mathbf{Z}/p$ is trivial. $\square$

Proof of Proposition 5.5. Let k be any positive integer. Lemma 5.6 implies that $\mathrm{Res}(g, t^{p^k} - 1) \neq 0$. Hence β_{p^k} vanishes, and therefore $\widetilde{\beta}_{p^k}$ is also zero. By a result analagous to Proposition 2.5 and the multiplicative property of resultants

$$\widetilde{b}_{p^k} = \left|\mathrm{Res}(\Delta, \nu_{p^k})\right| = \left|\mathrm{Res}(t-1, \nu_{p^k})\right|^q \left|\mathrm{Res}(g, \nu_{p^k})\right| = (p^k)^q \left|\mathrm{Res}(g, \nu_{p^k})\right|.$$

By Lemma 5.6, p does not divide $\left|\mathrm{Res}(g, t^{p^k} - 1)\right|$. Hence p does not divide $\mathrm{Res}(g, \nu_{p^k})$, and so $b^{(p)}_{p^k} = p^{kq}$. $\square$

COROLLARY 5.7. (i) *Let M_r be the r-fold cyclic cover of S^3 branched over a knot. If r is a prime power p^k, then the p-torsion submodule of $H_1(M_r;\mathbf{Z})$ is trivial.*

(ii) *Let M_r be the r-fold cyclic cover S^3 branched over a 2-component link $l = l_1 \cup l_2$. If r is a power of a prime that does not divide* $\mathrm{Lk}(l_1, l_2)$, *then the p-torsion submodule of $H_1(M_r;\mathbf{Z})$ is trivial.*

Proof. Statement (i) was proven in [Go78]. Here it follows from Proposition 5.5 together with the well-known fact that $|\Delta(1)| = 1$, whenever Δ is the Alexander polynomial of a knot. The second statement is a consequence of Proposition 5.5 and Remark 5.4 (ii). □

PROPOSITION 5.8. *Suppose that $\mathcal{M}$ is a finitely generated $\mathcal{R}_1$-module that is isomorphic to $\mathcal{R}_1/(\Delta)$. If $\Delta(t) = (t-1)^q g(t)$, where $g(1) \neq 0$, then for every positive integer r, there exists a positive integer δ'_r such that*

$$\widetilde{b}_r = (\delta'_r)^q \cdot |T(\mathcal{R}_1/(g, \nu_r))| \,.$$

Moreover, $\delta'_{r+\gamma} = \delta'_r$, for all r, where γ is the cyclotomic order of Δ.

REMARKS 5.9.

(i) The order $|T(\mathcal{R}_1/(g, \nu_r))|$ can be found using Proposition 5.3 and Theorem 3.3.

(ii) When $\mathcal{M}$ is a direct sum of cyclic modules, $\widetilde{b}_r$ can again be found by applying Proposition 5.5 to each summand. When $\mathcal{M}$ is not a direct sum of cyclic modules but is torsion free as an abelian group, a result analogous to Theorem 3.6 can be found by replacing $t^r - 1$ everywhere by ν_r in the proof. As in Section 3, the torsion numbers $\widetilde{b}_r$ are then seen to satisfy a linear homogeneous recurrence relation.

Proof of Proposition 5.8. Consider the exact sequence

$$0 \to \ker g \to \mathcal{R}_1/((t-1)^q, \nu_r) \xrightarrow{g} \mathcal{R}_1/((t-1)^q g, \nu_r) \xrightarrow{\pi} \mathcal{R}_1/(g, \nu_r) \to 0\,,$$

where the first homomorphism is inclusion, the second is multiplication by g, and the third is the natural projection. The order of $\mathcal{R}_1/((t-1)^q, \nu_r)$ is equal to $|\mathrm{Res}((t-1)^q, \nu_r)|$, which is equal to r^q. The kernel of g is generated by ν_r/f_r, where f_r is the greatest common divisor of g and ν_r. Notice that $f_{r+\gamma} = f_r$, for all r. Lemmas 5.2 and 3.7 complete the proof. □

We conclude with a generalization of Corollary 5.7 (ii).

When (G, χ) is the augmented group corresponding to a 2-component link l, the epimorphism χ factors through $\eta : G \to G_{ab} \cong \mathbf{Z}^2$. For any finite-index subgroup $\Lambda \subset \mathbf{Z}^2$ there is a $|\mathbf{Z}^2/\Lambda|$-fold cover of S^3 branched over l corresponding to the map $G \to \mathbf{Z}^2 \to \mathbf{Z}^2/\Lambda$. The cover M_r is a special case corresponding to the subgroup Λ generated by $t_1 - t_2$, t_1^r, t_2^r. We denote the rank of $H_1(M_\Lambda; \mathbf{Z})$ by β_Λ and the order $|TH_1(M_\Lambda; \mathbf{Z})|$ by b_Λ.

THEOREM 5.10. *Let $l = l_1 \cup l_2$ be a link in S^3. If p is a prime that does not divide* $\mathrm{Lk}(l_1, l_2)$, *then $\beta_\Lambda = 0$ and $b_\Lambda^{(p)} = 1$ for any subgroup $\Lambda \subset \mathbf{Z}^2$ of index p^k, $k \geq 1$.*

Proof. Let $\mathcal{M}_\eta$ be the kernel of η. We consider the dual $\mathcal{M}_\eta^\wedge$, which is a compact abelian group with a $\mathbf{Z}^2$-action by automorphisms induced by conjugation in G by t_1 and t_2. The automorphism induced by $\mathbf{n} \in \mathbf{Z}^2$ is denoted by $\sigma_\mathbf{n}$; the automorphims induced by $(1,0)$ and $(0,1)$ are abbreviated by σ_1 and σ_2, respectively. The dual $\mathcal{M}_\eta^\wedge$ can be identified with a subspace of $\mathrm{Fix}_\Lambda(\sigma) = \{\rho \in \mathcal{M}_\eta^\wedge \ : \ \sigma_\mathbf{n}\rho = \rho \text{ for all } \mathbf{n} \in \Lambda\}$. Details can be found in [SW00].

From the elementary ideals of $\mathcal{M}_\eta$ a sequence of 2-variable Alexander polynomials $\Delta_i(t_1, t_2)$ is defined; when $i = 0$, setting $t_1 = t_2 = t$ recovers $\Delta(t)$. By [Cr65], $\Delta_0(t_1, t_2)$ annihilates $\mathcal{M}_\eta$. Hence $\Delta_0(\sigma_1, \sigma_2)\rho = 0$ for all $\rho \in \mathcal{M}_\eta^\wedge$. Consequently, if $\sigma_\mathbf{n}\rho = \rho$ for all $\mathbf{n} \in \mathbf{Z}^2$ then $0 = \Delta_0(\sigma_1, \sigma_2)\rho = \Delta_0(1,1)\rho = \Delta(1)\rho$. Recall that $\Delta(1) = \mathrm{Lk}(l_1, l_2)$.

Let

$$Y = \{\rho\colon \mathcal{M}_\eta \to \mathbf{Z}/p \ : \ \sigma_\mathbf{n}\rho = \rho \text{ for all } \mathbf{n} \in \Lambda\}.$$

We identify $\mathbf{Z}/p$ with the group of p^{th} roots of unity, so that Y is contained in $\mathcal{M}_\eta^\wedge$. It is a subspace of $\mathrm{Fix}_\Lambda(\sigma)$ invariant under the $\mathbf{Z}^2$-action, and it contains a subspace isomorphic to $\mathcal{M}_\eta \otimes_\mathbf{Z} \mathbf{Z}/p$. It suffices to prove that Y is trivial.

Our hypothesis that p does not divide the linking number of l_1 and l_2 implies that $\Delta_0(t_1, t_2)$ is not zero. Consequently, Y is a finite p-group and so its order is a power of p. In view of the second paragraph, the hypothesis also implies that the only point fixed by the $\mathbf{Z}^2$-action is trivial. But

$$|Y| = \sum |\mathcal{O}_\rho| = \sum |\mathbf{Z}^d/\operatorname{stab}(\rho)|,$$

where the sums are taken over distinct orbits $\mathcal{O}_\rho$ and stabilizers $\operatorname{stab}(\rho)$, respectively. Each stabilizer contains Λ, and so $|\mathbf{Z}^d/\operatorname{stab}(\rho)|$ is a divisor of p^k whenever $\rho \neq 0$. Hence $|Y|$ is congruent to 1 mod p. Since $|Y|$ is a power of p, the subspace Y must be trivial. □