# 1. Introduction

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# PROJECTIVE MODULES OVER SOME PRÜFER RINGS

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## 1. INTRODUCTION

This note is a contribution to the theory of Prüfer rings with 0-divisors. For certain classes of Prüfer rings $R$, we classify the f. g. (finitely generated) projective $R$-modules, compute the reduced Grothendieck group $\widetilde{K}_0(R)$, and show that non countably generated projective $R$-modules are free.

All rings $R$ in this note will be assumed to be commutative with an identity element. The set of 0-divisors in $R$ is denoted by $\mathcal{Z}(R)$; for convenience, we take 0 to be an element of $\mathcal{Z}(R)$. Elements in $R \setminus \mathcal{Z}(R)$ (the non 0-divisors) are said to be *regular*; an ideal in $R$ is said to be regular if it contains a regular element of $R$. By definition, a Prüfer ring is a (commutative) ring in which every f. g. regular ideal is invertible. Many characterizations of Prüfer rings are known; see, e.g. [Gr] and [LM].

The theory of Prüfer *domains* is by now very well developed. For an excellent presentation of this theory, see the recent monograph by Fontana, Huckaba, and Papick [FHP]. Since Prüfer domains are precisely the semihereditary domains [1]), a classical theorem of Cartan and Eilenberg [CE: Ch. I, Prop. 6.1] implies that, over such rings, every f. g. projective module is isomorphic to a direct sum of (f. g. projective) ideals. Such a result is, however, not available over an arbitrary Prüfer ring $R$, since $R$ may no longer be semihereditary. In Theorem 3 of this note, we'll show that, by imposing a "smallness" condition on the set of 0-divisors $\mathcal{Z}(R)$, we can restore a decomposition theorem on the f. g. projective modules over a Prüfer ring $R$. By definition, a 0-divisor $a \in \mathcal{Z}(R)$ is *small* if $Ra$ is a small ideal, that is, for

[1]) A ring is called *hereditary* if all ideals are projective, and *semihereditary* if all f. g. ideals are projective.

any ideal $A \subseteq R$, $A + Ra = R$ implies that $A = R$. We say that $R$ *has small* $0$*-divisors* if all $a \in \mathcal{Z}(R)$ are small. (For instance, integral domains and local rings have small $0$-divisors.)

In the classical case of Dedekind domains, it is well known that the decomposition of f. g. projective modules into ideals is not unique, but the degree of nonuniqueness is completely controlled by the Steinitz Isomorphism Theorem; see, e.g. [$\text{Ka}_2$]. In [HL], Heitmann and Levy have obtained a generalization of the Steinitz Isomorphism Theorem to Prüfer domains having the so-called $1\frac{1}{2}$ generator property. As a consequence, the usual classification results for f. g. projective modules over Dedekind domains extend smoothly to this class of Prüfer domains. In order to further extend these results to rings with $0$-divisors, we make the following modification of the definition given by Heitmann and Levy: a ring $R$ is said to have the $1\frac{1}{2}$ *generator property* if, for any invertible ideal $I \subseteq R$ and any regular element $a \in I \setminus (\text{rad}\, R)I$, there exists an element $b$ such that $I = Ra + Rb$. (Here and in the following, $\text{rad}(R)$ denotes the Jacobson radical of the ring $R$.) In §2, generalizing the work in [HL], we prove the Steinitz Isomorphism Theorem for this class of rings, again by using a smallness assumption on $\mathcal{Z}(R)$.

In the context of Prüfer rings (and in the language of algebraic $K$-theory), the first principal result in this paper can be stated as follows.

THEOREM A. *Let $R$ be a Prüfer ring with the $1\frac{1}{2}$ generator property having small $0$-divisors. Then $\widetilde{K}_0(R) \cong \text{Pic}(R)$. (Here, $\widetilde{K}_0(R) := K_0(R)/\mathbf{Z}\cdot[R]$, and* Pic$(R)$ *denotes the Picard group of $R$.)*

The second principal result concerns projective modules that are not necessarily f. g. over Prüfer rings. Using some of the methods of Kaplansky [$\text{Ka}_2$, $\text{Ka}_3$] and Bass [Ba], we obtain the following.

THEOREM B. *Let $R$ be a Prüfer ring with small $0$-divisors. Then a projective $R$-module is indecomposable if and only if it is isomorphic to an invertible ideal of $R$. If, moreover, $R$ has the $1\frac{1}{2}$ generator property, then any infinite direct sum of nonzero countably generated projective $R$-modules is free, and any non countably generated projective $R$-module is free.*

In the course of proving these results, we have also generalized several known theorems in the literature from domains to arbitrary (commutative) rings; see §4.