3. Hyperbolic metric spaces

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3. Hyperbolic metric spaces

We refer the reader to [1, 4, 14, 20, 25, 32, 40] for the basic information about Gromov-hyperbolic metric spaces. We briefly recall the main definitions.

If (X,d) is a geodesic metric space and $x, y \in X$, we shall denote by $[x,y]$ a geodesic segment from x to y in X.

Definition 3.1 (Gromov product). Let (X,d) be a metric space and suppose $x, y, z \in X$. We set

$$(x,y)_z := \frac{1}{2}[d(z,x) + d(z,y) - d(x,y)].$$

Note that $(x,y)_z = (y,x)_z$.

Definition 3.2 (Hyperbolic metric space [1]). Let (X,d) be a geodesic metric space. We say that (X,d) is *δ-hyperbolic* (where $\delta \geq 0$) if for any $p, x, y, z \in X$ we have:

$$(x,y)_p \geq \min\{(x,z)_p, (y,z)_p\} - \delta.$$

The space X is said to be *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

There are many equivalent definitions of hyperbolicity, for example:

Proposition 3.3 ([1, 20, 32]). *Let (X,d) be a geodesic metric space. Then the following conditions are equivalent.*

1. *The space X is hyperbolic.*
2. *There exists a constant $\delta' \geq 0$ such that if $x, y, z \in X$ and $y' \in [x,y]$, $z' \in [x,z]$ are such that $d(x,y') = d(x,z') \leq (y,z)_x$ then $d(y',z') \leq \delta'$.*
3. (Thin Triangles Condition) *There exists $\delta'' \geq 0$ such that for any $x, y, z \in X$, for any geodesic segments $[x,y]$, $[x,z]$ and $[y,z]$ and for any point $p \in [x,y]$ there is a point $q \in [x,z] \cup [y,z]$ such that $d(p,q) \leq \delta''$.*

Definition 3.4 (Word-hyperbolic group). A finitely generated group G is said to be *word-hyperbolic* if for some (and hence for any) finite generating set A of G the Cayley graph $\Gamma(G,A)$ is hyperbolic.

Definition 3.5 (Gromov product for sets). Let (X,d) be a metric space. Let $x \in X$ and $Q, Q' \subseteq X$. Define $(Q,Q')_x := \sup\{(q,q')_x \mid q \in Q,\ q' \in Q'\}$.