Introduction

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A TRIPLE RATIO ON THE UNITARY STIEFEL MANIFOLD

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ABSTRACT. For the unitary Stiefel manifold S realized as the Shilov boundary of the unit ball D in $\mathrm{Mat}(p \times q, \mathbf{C})$, we construct characteristic invariants for the (generic) orbits of the conformal group $\mathbf{PSU}(p,q)$ in $S \times S \times S$. The construction uses the automorphy kernel of the bounded symmetric domain.

INTRODUCTION

Let $D = G/K$ be a bounded symmetric domain in a complex vector space $\mathbf{C}^N$, and let S be its Shilov boundary. The action of G extends to S and this action is transitive on S. It is generally referred to in the literature as the *conformal action* of G on S. One can show that the action is almost 2-transitive in the sense that G has a dense open orbit in $S \times S$. Hence it is a natural question to look for the G-orbits in $S \times S \times S$ and for characteristic invariants of this action. If D happens to be of tube type (in which case $\dim_{\mathbf{R}} S = \dim_{\mathbf{C}} D$), this question was solved in [CØ]. There are a finite number of open orbits in $S \times S \times S$, and the (generalized) *Maslov index* we constructed is a characteristic invariant for the G-action. In the case of the unit ball in $\mathbf{C}^2$, the Shilov boundary coincides with the topological boundary, namely the unit sphere $S = \mathbf{S}^3$. In [Ca], E. Cartan constructed a (real-valued) invariant for triples on S (he called S the "hypersphere"). Independently (and more than 50 years later) Korányi and Reimann studied the case of the unit ball in $\mathbf{C}^n$ (see [KR]). Through the Cayley transform, the problem is changed into an equivalent problem for the Heisenberg group $\mathbf{H}_n$ under the action of its conformal group $G = \mathbf{PSU}(n+1,1)$. For this situation, they studied a complex cross ratio on $\mathbf{H}_n$, from which they were able (in a rather indirect way) to construct a (real-valued) invariant for triples, which characterizes the G-orbits of triples in $\mathbf{H}_n$. Here we solve the problem for the case where D

is the unit ball in the matrix space $\mathrm{Mat}(p \times q, \mathbf{C})$, S is the unitary Stiefel manifold $\mathbf{S}_{p,q}$ and $G = \mathbf{PSU}(p,q)$. The invariant we construct for triples is of matrix-valued nature (it is a conjugacy class) and we give two versions of it (see Theorems 4.3 and 4.4). The basic strategy is to approach the Shilov boundary from inside. The (matrix-valued) *automorphy kernel* for the domain D is used to build a kernel for triples of points inside D which transforms nicely under the action of G. It remains to look carefully at the boundary behaviour of the kernel when the points approach the Shilov boundary S. This is only possible for triples satisfying a generic condition called *tranversality* (see Proposition 2.1 for a definition). The *Cayley transform* plays an important role in the proofs. Finally the problem is reduced to a *linear* problem, which is related to the description of some orbits for the action $(g,X) \mapsto gXg^*$ of GL_q on $\mathrm{Mat}(q \times q, \mathbf{C})$ (see Theorem 3.9).

For general references on bounded symmetric domains and their geometric properties, see [S], and Part III in [Fal]. For explicit calculations related to our example, see [P] and [H].

1. Geometric setting

Let p, q be two integers with $1 \leq q \leq p$, and let

$$D = \{z \in \mathrm{Mat}(p \times q, \mathbf{C}) \mid \mathbf{1}_q - z^* z \gg 0\}\,. \tag{1}$$

Let $G = \mathrm{SU}(p,q) \subset \mathrm{GL}(p+q, \mathbf{C})$. An element $g \in \mathrm{GL}(p+q, \mathbf{C})$ will often be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a \in \mathrm{Mat}(p \times p, \mathbf{C}),\ b \in \mathrm{Mat}(p \times q, \mathbf{C}),\ c \in \mathrm{Mat}(q \times p, \mathbf{C}),\ d \in \mathrm{Mat}(q \times q, \mathbf{C})\,.$$

In this notation, the conditions for g to belong to $\mathrm{U}(p,q,\mathbf{C})$ can be written as

$$\begin{aligned} a^*a - c^*c &= \mathbf{1}_p \\ b^*a - d^*c &= 0 \\ d^*d - b^*b &= \mathbf{1}_q\,. \end{aligned} \tag{2}$$

Define an action of the group $\mathrm{GL}(p+q, \mathbf{C})$ on $\mathrm{Mat}(p \times q, \mathbf{C})$ by

$$g(z) = (az+b)(cz+d)^{-1}\,. \tag{3}$$