

1. Geometric setting

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is the unit ball in the matrix space $\text{Mat}(p \times q, \mathbf{C})$, S is the unitary Stiefel manifold $\mathbf{S}_{p,q}$ and $G = \mathbf{PSU}(p, q)$. The invariant we construct for triples is of matrix-valued nature (it is a conjugacy class) and we give two versions of it (see Theorems 4.3 and 4.4). The basic strategy is to approach the Shilov boundary from inside. The (matrix-valued) *automorphy kernel* for the domain D is used to build a kernel for triples of points inside D which transforms nicely under the action of G . It remains to look carefully at the boundary behaviour of the kernel when the points approach the Shilov boundary S . This is only possible for triples satisfying a generic condition called *transversality* (see Proposition 2.1 for a definition). The *Cayley transform* plays an important role in the proofs. Finally the problem is reduced to a *linear* problem, which is related to the description of some orbits for the action $(g, X) \mapsto gXg^*$ of GL_q on $\text{Mat}(q \times q, \mathbf{C})$ (see Theorem 3.9).

For general references on bounded symmetric domains and their geometric properties, see [S], and Part III in [Fal]. For explicit calculations related to our example, see [P] and [H].

1. GEOMETRIC SETTING

Let p, q be two integers with $1 \leq q \leq p$, and let

$$(1) \quad D = \{z \in \text{Mat}(p \times q, \mathbf{C}) \mid \mathbf{1}_q - z^*z \gg 0\}.$$

Let $G = \mathbf{SU}(p, q) \subset \text{GL}(p + q, \mathbf{C})$. An element $g \in \text{GL}(p + q, \mathbf{C})$ will often be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a \in \text{Mat}(p \times p, \mathbf{C}), \quad b \in \text{Mat}(p \times q, \mathbf{C}), \quad c \in \text{Mat}(q \times p, \mathbf{C}), \quad d \in \text{Mat}(q \times q, \mathbf{C}).$$

In this notation, the conditions for g to belong to $\mathbf{U}(p, q, \mathbf{C})$ can be written as

$$(2) \quad \begin{aligned} a^*a - c^*c &= \mathbf{1}_p \\ b^*a - d^*c &= 0 \\ d^*d - b^*b &= \mathbf{1}_q. \end{aligned}$$

Define an action of the group $\text{GL}(p + q, \mathbf{C})$ on $\text{Mat}(p \times q, \mathbf{C})$ by

$$(3) \quad g(z) = (az + b)(cz + d)^{-1}.$$

The action is not everywhere defined, but it is certainly defined if $g \in G$ and $z \in D$. It defines an action of G on D , and G (or rather $\text{PSU}(p, q)$) is the neutral component of the group of all biholomorphic transformations of D .

The stabilizer of the base point $0 \in D$ is the maximal compact subgroup $K = S(\text{U}(p) \times \text{U}(q))$. Its complexification is the complex group $K^{\mathbf{C}} = S(\text{GL}(p, \mathbf{C}) \times \text{GL}(q, \mathbf{C}))$. We also define the following subgroups

$$P^+ = \left\{ \begin{pmatrix} \mathbf{1}_p & z \\ 0 & \mathbf{1}_q \end{pmatrix}, z \in \text{Mat}(p \times q, \mathbf{C}) \right\}$$

$$P^- = \left\{ \begin{pmatrix} \mathbf{1}_p & 0 \\ w & \mathbf{1}_q \end{pmatrix}, w \in \text{Mat}(q \times p, \mathbf{C}) \right\}.$$

The corresponding *Harish Chandra decomposition* is the following identity

$$(4) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1}_p & bd^{-1} \\ 0 & \mathbf{1}_q \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ d^{-1}c & \mathbf{1}_q \end{pmatrix}$$

valid for $g \in \text{GL}(p + q, \mathbf{C})$ if d is invertible.

The *automorphy kernel* $k(z, w)$ is defined for $z, w \in \text{Mat}(p \times q, \mathbf{C})$ wherever it makes sense by the formula

$$(5) \quad k(z, w) = (\mathbf{1}_q - w^*z)^{-1}.$$

In particular it is always well defined for $z, w \in D$ and has values in $\text{GL}(q, \mathbf{C})$. It has the following law of transformation for $g \in G$

$$(6) \quad k(g(z), g(w)) = j(g, z)k(z, w)j(g, w)^*,$$

where

$$(7) \quad j(g, z) = cz + d.$$

The *Shilov boundary* of D is the unitary Stiefel manifold S defined by

$$(8) \quad S = \{ \sigma \in \text{Mat}(p \times q, \mathbf{C}) \mid \sigma^* \sigma = \mathbf{1}_q \}.$$

The action of G extends to S , and it is clearly transitive on S . In fact the action of K is already transitive.

To go further, we need to make a specific choice of a base point in S . For this we first systematically write elements in $\text{Mat}(p \times q, \mathbf{C})$ as

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix}$$

where $z_q \in \text{Mat}(q \times q, \mathbf{C})$ and $z' \in \text{Mat}((p - q) \times q, \mathbf{C})$. With this convention, let $ie = \begin{pmatrix} i\mathbf{1}_q \\ 0 \end{pmatrix}$ be the base point in S . Associated to this choice is the Cayley transform c , given by

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix} \mapsto c(z) = \begin{pmatrix} w_q \\ w' \end{pmatrix}$$

with

$$(9) \quad \begin{aligned} w_q &= (z_q + i\mathbf{1}_q)(iz_q + \mathbf{1}_q)^{-1} \\ w' &= -z'(iz_q + \mathbf{1}_q)^{-1}. \end{aligned}$$

The inverse of the Cayley transform is the map which to $\begin{pmatrix} w_q \\ w' \end{pmatrix}$ associates the matrix $\begin{pmatrix} z_q \\ z' \end{pmatrix}$ given by

$$(10) \quad \begin{aligned} z_q &= (iw_q - \mathbf{1}_q)^{-1}(i\mathbf{1}_q - w_q) \\ z' &= 2w'(iw_q - \mathbf{1}_q)^{-1}. \end{aligned}$$

The Cayley transform is a rational map, well defined on D . The image of D is the *Siegel domain of type II* defined by

$$(11) \quad {}^cD = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix}, \frac{1}{2i}(w_q - w_q^*) - w'^*w' \gg 0 \right\}$$

and the image of the Shilov boundary (more exactly the part of the Shilov boundary where the Cayley transform is defined) is

$$(12) \quad {}^cS = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix}, \frac{1}{2i}(w_q - w_q^*) = w'^*w' \right\}.$$

To the data

$$w_0 \in \text{Mat}((p - q) \times q, \mathbf{C})$$

$$h \in \text{GL}(q, \mathbf{C}), u \in \text{U}(p - q, \mathbf{C}), \text{ such that } \det h = (\det u)^{-1}$$

$$s \in \text{Herm}(q, \mathbf{C})$$

we associate the transform

$$(13) \quad \begin{aligned} w_q &\mapsto h^*w_qh + s + 2iw_0^*uw'h + iw_0^*w_0 \\ w' &\mapsto uw'h + w_0. \end{aligned}$$

Any such transform maps cD in a one-to-one fashion into itself. These transforms form a group and it is exactly the group of affine holomorphic transforms of the domain cD .

Let B be the stabilizer of the point ie in G . The conjugate group under the Cayley transform is ${}^cB = c \circ B \circ c^{-1}$ and it turns out to be exactly the group of affine transforms of cD we just described. Observe that the group cB is transitive on cD and on cS .