# 3. Orbits for the $GL_q$-action on $\tilde{T}_q$

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## 3. Orbits for the $\mathrm{GL}_q$-action on $\widetilde{T}_q$

Any $z \in \mathrm{Mat}(q \times q, \mathbf{C})$ can be written in a unique way as $z = x + iy$, with $x, y \in H_q$. We will be concerned with the set $\widetilde{T}_q$ defined by

(16) $\quad \widetilde{T}_q = \{z \in \mathrm{Mat}(q \times q, \mathbf{C}) \mid z = x + iy,\ x \in H_q,\ y \in \overline{\Omega}_q,\ \det z \neq 0\}.$

Its interior is the classical *tube domain* over the cone $\Omega_q$, namely

$$T_q = \{z \in \mathrm{Mat}(q \times q, \mathbf{C}) \mid z = x + iy,\ y \in \Omega_q\}.$$

Let $G = \mathrm{GL}(q, \mathbf{C})$ act on $\mathrm{Mat}(q \times q, \mathbf{C})$ by

(17) $$(g, z) \longmapsto gzg^*.$$

The spaces $H_q, \Omega_q, \overline{\Omega}_q$ are stable under this action, and hence $\widetilde{T}_q$ and $T_q$ are invariant subsets under this action. We investigate the orbits and describe a full set of invariants for this action.

There is a natural invariant associated to a $\mathrm{GL}(q, \mathbf{C})$-orbit. To any $z \in \widetilde{T}_q$, we associate its *angular matrix* defined by

(18) $$a = a(z) = {z^*}^{-1} z.$$

Then the matrix associated to $gzg^*$ is ${g^*}^{-1} a g^*$, so that the angular matrix $a(z)$ belongs to the same conjugacy class when $z$ runs through a $\mathrm{GL}(q, \mathbf{C})$-orbit. As we shall see (Theorem 3.3 and Theorem 3.13), this invariant is close to characterizing the orbits.

Let us first prove some elementary properties of the angular matrix.

PROPOSITION 3.1. *Let $z = x + iy \in \widetilde{T}_q$, and let $a = {z^*}^{-1} z$ be its angular matrix. Then*

(i) $\mathrm{Sp}(a) \subset \mathrm{U}_1 = \{\mu \in \mathbf{C}, |\mu| = 1\}$;

(ii) *if* $1 \in \mathrm{Sp}(a)$, *then* $y$ *is degenerate and*

$$\{v \in \mathbf{C}^q \mid av = v\} = \{v \in \mathbf{C}^q \mid yv = 0\}.$$

*Proof.* Let $\mu$ be an eigenvalue of $a$, and let $v \neq 0$ be an eigenvector for the eigenvalue $\mu$. Then $zv = \mu z^* v$, and hence

$$(zv, v) = \mu(z^* v, v) = \mu(v, zv) = \mu\,\overline{(zv, v)}.$$

If $(zv, v) \neq 0$, then $|\mu| = 1$. So we now assume $(zv, v) = 0$. This amounts to $(xv, v) + i(yv, v) = 0$, so that in particular $(yv, v) = 0$. Now recall that $y$ is positive semi-definite. So the condition $(yv, v) = 0$ implies that $yv = 0$. From this it follows that $zv = xv = z^* v$, and as $z$ is assumed to be invertible, this implies $\mu = 1$. This shows (i) and part of (ii). Conversely, the condition $yv = 0$ implies trivially $av = v$. $\square$

In particular, we may consider the polynomial $d(\mu) = \det(z - \mu z^*)$. The roots of $d$ are the eigenvalues of the angular matrix. The set of these roots, counted with their multiplicities, will be called the *angular spectrum* of $z$.

We first consider the case of $T_q$. So let $z = x + iy \in T_q$. Then as $y$ is positive-definite, we may define its square root $y^{1/2}$ as the unique positive-definite Hermitian matrix whose square is $y$. Then we may write

$$x + iy = y^{\frac{1}{2}}(y^{-\frac{1}{2}}xy^{-\frac{1}{2}} + i1_q)y^{\frac{1}{2}} .$$

This shows that any $\mathrm{GL}(q, \mathbf{C})$-orbit contains some element of the form $x + i1_q$, where $x \in H_q$. But by the classical diagonalization theorem for Hermitian forms, there exists an orthonormal basis in which the Hermitian form associated to $x$ is diagonal. In other words, there exists a unitary matrix $u$ and real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$ such that

$$uxu^* = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix} .$$

Moreover, if $\Lambda$ and $\Lambda'$ are two such diagonal matrices, then $\Lambda + i\mathbf{1}_q$ and $\Lambda' + i\mathbf{1}_q$ are not conjugate under the action of $\mathrm{GL}(q, \mathbf{C})$ unless $\Lambda = \Lambda'$. Hence we have shown the following result, which of course is the well-known fact that there is a simultaneous diagonalization for two Hermitian forms if one of them is positive-definite.

THEOREM 3.2. *The set of matrices of the form*

$$\Lambda = \begin{pmatrix} \lambda_1 + i & & & \\ & \lambda_2 + i & & \\ & & \ddots & \\ & & & \lambda_q + i \end{pmatrix} \tag{19}$$

*with* $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_q$ *is a full set of representatives of the* $\mathrm{GL}(q, \mathbf{C})$*-orbits in* $T_q$.

The angular matrix associated to $\Lambda$ is

$$\begin{pmatrix} \frac{\lambda_1+i}{\lambda_1-i} & & & \\ & \frac{\lambda_2+i}{\lambda_2-i} & & \\ & & \ddots & \\ & & & \frac{\lambda_q+i}{\lambda_q-i} \end{pmatrix} . \tag{20}$$

The latter is a semi-simple matrix with spectral values

$$\mu_j = \frac{\lambda_j + i}{\lambda_j - i}$$

for $1 \leq j \leq q$. Observe that these spectral values are complex numbers of modulus $1$, but always different from $1$. From the $u_j$ we may recover the $\lambda_j$ by the formula

$$\lambda_j = i\,\frac{1+\mu_j}{1-\mu_j}\,.$$

From these observations we get the following result.

THEOREM 3.3. *Two elements $z$ and $z'$ of $T_q$ belong to the same* $\mathrm{GL}(q, \mathbf{C})$*-orbit if and only if their angular matrices are conjugate. The angular spectrum is a full set of invariants for the action of* $\mathrm{GL}(q, \mathbf{C})$ *on $T_q$.*

The situation for $\widetilde{T}_q$ is more complicated. In fact we may consider the extreme case where $y = 0$. Then $x$ corresponds to a non-degenerate Hermitian form, and the orbit picture is given by the signature. So we need to consider matrices of the form

$$\Upsilon = \Upsilon_{n_+, n_-} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

with $n_+$ diagonal entries equal to $+1$ and $n_-$ diagonal entries equal to $-1$, $n_+$ and $n_-$ being arbitrary nonnegative integers such that $n_+ + n_- = q$. The corresponding angular matrix is the identity matrix $\mathbf{1}_q$.

Another source of difficulty comes from the fact that it is not always possible to find a basis in which both Hermitian forms associated to $x$ and $y$ are diagonal. For instance if $q = 2$, consider the matrix

$$z = \begin{pmatrix} i & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice that its angular matrix is

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which is not semisimple.

From these examples we see that neither the angular spectrum of $z$ nor the conjugacy class of the angular matrix characterizes the orbit of $z$.

Let $n_1, n_2, n_3, n_4$ be four nonnegative integers such that $n_1+2n_2+n_3+n_4 = q$, and let $\lambda_1, \lambda_2, \dots, \lambda_{n_1}$ be $n_1$ real numbers satisfying the condition

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1} .$$

To such data we associate the matrix $\Lambda = \Lambda(\lambda_1, \lambda_2, \dots, \lambda_{n_1}, n_2, n_3, n_4)$ given by

$$\begin{pmatrix} \lambda_1 + i &&&&&&&&&&&& \\ & \ddots &&&&&&&&&&& \\ && \lambda_{n_1} + i &&&&&&&&&& \\ &&& i & 1 &&&&&&&& \\ &&& 1 & 0 &&&&&&&& \\ &&&&& \ddots &&&&&&& \\ &&&&&& i & 1 &&&&& \\ &&&&&& 1 & 0 &&&&& \\ &&&&&&&& 1 &&&& \\ &&&&&&&&& \ddots &&& \\ &&&&&&&&&& 1 && \\ &&&&&&&&&&& -1 & \\ &&&&&&&&&&&& \ddots \\ &&&&&&&&&&&&& -1 \end{pmatrix} \tag{21}$$

where there are $n_2$ diagonal $2 \times 2$ submatrices of the form $\begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}$, $n_3$ diagonal terms equal to $1$ and $n_4$ diagonal terms equal to $-1$.

THEOREM 3.4. *Any* $\mathrm{GL}(q, \mathbf{C})$ *orbit in* $\widetilde{T}_q$ *contains one and only one matrix of the form* $\Lambda(\lambda_1, \lambda_2, \dots, \lambda_{n_1}, n_2, n_3, n_4)$.

Before beginning the proof, let us prove a couple of lemmas. Lemmas 3.6 and 3.7 are related to the classical Gauss's algorithm for diagonalizing an Hermitian form. Let $r, s, n$ be three nonnegative integers such that $r + s = n$.

LEMMA 3.5. *The stabilizer in* $\mathrm{GL}(n, \mathbf{C})$ *of the matrix* $y_r = \begin{pmatrix} \mathbf{1}_r & \\ & \mathbf{0}_s \end{pmatrix}$ *is the subgroup*

$$G_r = \left\{ \begin{pmatrix} u & v \\ 0 & h \end{pmatrix} \right\} \tag{22}$$

*where* $u \in \mathrm{U}(r)$, $v \in \mathrm{Mat}(r, s)$, $h \in \mathrm{GL}(s, \mathbf{C})$.

*Proof.* Easy computation.

Now we study the action of $G_r$ in $H_n$. If $x \in H_n$, let us write

$$x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix}$$

where $\alpha \in H_r, b \in \mathrm{Mat}(r \times s, \mathbf{C})$ and $\gamma \in H_s$. If $g = \begin{pmatrix} u & v \\ 0 & h \end{pmatrix} \in G_r$, then $gxg^* = \begin{pmatrix} \alpha' & b' \\ b'^* & \gamma' \end{pmatrix}$, with

$$\alpha' = u\alpha u^* + ubv^* + vb^*u^* + v\gamma v^*$$
$$b' = ubh^* + v\gamma h^*$$
$$\gamma' = h\gamma h^* .$$

LEMMA 3.6. *Let* $x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix} \in H_n$, *with* $\alpha \in H_r$, $b \in \mathrm{Mat}(r \times s, \mathbf{C})$ *and* $\gamma \in H_s$ . *Assume* $\det\gamma \neq 0$. *Then the orbit of* $x$ *under* $G_r$ *contains a matrix of the form* $\begin{pmatrix} \alpha' & 0 \\ 0 & \gamma \end{pmatrix}$ *with* $\alpha' \in H_r$.

*Proof.* This is a consequence of the previous formula with $u = \mathbf{1}_r$, $v = -b\gamma^{-1}$ and $h = \mathbf{1}_s$.

LEMMA 3.7. *Let* $x = \begin{pmatrix} \alpha & b \\ b^* & 0 \end{pmatrix} \in H_n$, *with* $\operatorname{rank} b = s$ *(so in particular* $r \geq s$*). Then the orbit of* $x$ *under* $G_r$ *contains an element of the form*

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

*with* $\beta \in H_{r-s}$.

*Proof.* Consider the subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & h \end{pmatrix}, \ u \in \mathrm{U}(r), \ h \in \mathrm{GL}_s(\mathbf{C}) \right\}$. It acts on the component $b$ by $b' = ubh^*$. As $\operatorname{rank}(b) = s$, we may think of $b$ as a set of $s$ independent vectors in $\mathbf{C}^r$. By the Gram-Schmidt process, it is possible to find $h \in \mathrm{GL}_s(\mathbf{C})$ such that $bh^*$ is a $s$-orthonormal frame in $\mathbf{C}^r$. But now two such frames are conjugate by the (left) action of $\mathrm{U}(r)$. Hence there exists $u \in \mathrm{U}(r)$ such that

$$ubh^* = \begin{pmatrix} 0 \\ \mathbf{1}_s \end{pmatrix} .$$

The matrix $x$ we started with is conjugate under $G_r$ to a matrix of the form

$$\begin{pmatrix} \alpha' & c & 0 \\ c^* & \beta & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

where $\alpha' \in H_{r-s}$, $\beta \in H_s$ and $c \in \mathrm{Mat}((r-s)\times s, \mathbf{C})$. Now we use the action of the element

$$g = \begin{pmatrix} \mathbf{1}_{r-s} & 0 & -c \\ 0 & \mathbf{1}_s & -\frac{\beta}{2} \\ 0 & 0 & \mathbf{1}_s \end{pmatrix} \in G_r$$

to get the result. $\square$

We are now ready to start the proof of Theorem 3.4.

STEP 1. Let $z = x + iy \in \widetilde{T}_q$. As $y$ is positive semidefinite, there exists an element $g \in \mathrm{GL}(q, \mathbf{C})$ such that

$$gyg^* = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

with $r$ diagonal entries equal to $1$, and $s$ diagonal entries equal to $0$, $r$ and $s$ being nonnegative integers satisfying $r + s = q$. In other terms, any $\mathrm{GL}(q, \mathbf{C})$-orbit in $\widetilde{T}_q$ contains an element of the form

$$\begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}$$

with $\alpha \in H_r, \gamma \in H_s, b \in \mathrm{Mat}(r \times s, \mathbf{C})$.

STEP 2. Now assume $x$ is of the form

$$x = \begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}.$$

Consider $\gamma$. It is an Hermitian matrix of size $s$, and under the action of $\mathrm{GL}(s, \mathbf{C})$ it can be transformed to

$$\begin{pmatrix} \mathbf{0}_{n_2} & 0 & 0 \\ 0 & \mathbf{1}_{n_3} & 0 \\ 0 & 0 & -\mathbf{1}_{n_4} \end{pmatrix}$$

where $n_2+n_3+n_4=s$. Hence $x$ is conjugate under the action of $G_r$ to an element of the form

$$\begin{pmatrix} \alpha & b' & c' \\ b'^* & 0 & 0 \\ c'^* & 0 & \Upsilon \end{pmatrix}$$

where $\alpha \in H_r$, $b' \in \mathrm{Mat}(r \times n_2, \mathbf{C})$, $c' \in \mathrm{Mat}(r \times (n_3+n_4), \mathbf{C})$ and

$$\Upsilon = \begin{pmatrix} \mathbf{1}_{n_3} & 0 \\ 0 & -\mathbf{1}_{n_4} \end{pmatrix}.$$

Using Lemma 3.6, we see that $x$ is conjugate under the action of $G_s$ to an element of the form

$$\begin{pmatrix} \alpha'' & b'' & 0 \\ b''^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix},$$

with $\alpha'' \in H_r$, $b'' \in \mathrm{Mat}(r \times n_2, \mathbf{C})$.

STEP 3. Assume now that

$$x = \begin{pmatrix} \alpha & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

with $\alpha \in H_r$ and $b \in \mathrm{Mat}(r \times n_2, \mathbf{C})$. Recall that

$$x+iy = \begin{pmatrix} \alpha + i\mathbf{1}_r & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

is assumed to be invertible. This shows that $\mathrm{rank}(b) = n_2$. So we may apply Lemma 3.7 to see that $x$ is conjugate under $G_r$ to an element of the form

$$\begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_2} & 0 \\ 0 & \mathbf{1}_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \Upsilon \end{pmatrix}$$

with $\beta \in H_{r-n_2}$.

STEP 4. Set $n_1 = r - n_2$. The last step is just to put the element $\beta \in H_{n_1}$ in diagonal form under the action of $\mathrm{U}(n_1)$. Up to minor rearrangements of the matrix, this shows that any $\mathrm{GL}(q,\mathbf{C})$-orbit in $\widetilde{T}_q$ contains an element of the form $\Lambda(\lambda_1, \ldots, \lambda_{n_1}, n_2, n_3, n_4)$.

STEP 5. It remains to show that two $\Lambda$'s are not conjugate under $\mathrm{GL}(q,\mathbf{C})$. The angular matrix associated to $\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)$ is

$$\begin{pmatrix} \frac{\lambda_1+i}{\lambda_1-i} &&&&&&&&& \\ & \ddots &&&&&&&& \\ && \frac{\lambda_{n_1}+i}{\lambda_{n_1}-i} &&&&&&& \\ &&& 1 & 0 &&&&& \\ &&& 2i & 1 &&&&& \\ &&&&& \ddots &&&& \\ &&&&&& 1 & 0 && \\ &&&&&& 2i & 1 && \\ &&&&&&&& 1 & \\ &&&&&&&&& \ddots \\ &&&&&&&&&& 1 \end{pmatrix}$$

where there are $n_2$ $2\times 2$ submatrices $\begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}$, and $n_3+n_4$ diagonal elements equal to 1. From the Jordan normal form theorem, we deduce that if $\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)$ and $\Lambda(\lambda'_1,\dots,\lambda'_{n_1},n'_2,n'_3,n'_4)$ are in a same $\mathrm{GL}(q,\mathbf{C})$-orbit, then $n_1=n'_1$, $\lambda_j=\lambda'_j$ for all $j, 1\le j\le n_1$, $n_2=n'_2$ and $n_3+n_4=n'_3+n'_4$. Now the matrix $\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)=L+iM$ and $\Lambda'=L'+iM'$, with $L,L',M,M'\in H_n$. As $\Lambda$ and $\Lambda'$ are supposed to be in the same $\mathrm{GL}(q,\mathbf{C})$-orbit, $L$ and $L'$ are also in the same $\mathrm{GL}(q,\mathbf{C})$-orbit, and so they must have the same signature. This forces $n_3=n'_3$ and $n_4=n'_4$, and hence $\Lambda=\Lambda'$.

We can now give the solution to the orbit problem we addressed at the end of Section 2. Recall that for any integer $r$ such that $0\le r\le q$ we defined

$$\widetilde{T}_q^{(r)}=\{z=x+iy \mid y\in\overline{\Omega}_q,\ \operatorname{rank}(y)\le r,\ z \text{ invertible}\}.$$

LEMMA 3.8. *Let $n_1,n_2,n_3,n_4$ be four integers such that*

$$n_1+2n_2+n_3+n_4=q,$$

*and let $\lambda_1,\dots,\lambda_{n_1}$ be $n_1$ real numbers. Then the standard matrix $\Lambda=\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)$ belongs to $\widetilde{T}_q^{(r)}$ if and only if $n_1+n_2\le r$.*

In fact the rank of $\frac{1}{2i}(\Lambda-\Lambda^*)$ is $n_1+n_2$.

THEOREM 3.9. *Any* $\mathrm{GL}(q,\mathbf{C})$*-orbit in* $\widetilde{T}_q^{(r)}$ *contains a unique standard matrix* $\Lambda((\lambda_1,\ldots,\lambda_{n_1},n_2,n_3,n_4)$ *with* $n_1+n_2\leq r$.

We now want an analog of Theorem 3.3. As we have already noticed, the conjugacy class of the angular matrix does not determine the orbit of the matrix. We need a finer invariant, which we will construct now.

LEMMA 3.10. *The space* $\widetilde{T}_q$ *is connected and simply connected.*

*Proof.* As $T_q$ is connected and $T_q\subset\widetilde{T}_q\subset\overline{T_q}$, the space $\widetilde{T}_q$ is connected. Take $i\mathbf{1}_q$ as base point in $\widetilde{T}_q$, and observe that for any $z\in\widetilde{T}_q$ and any $s>0$, $z+is\mathbf{1}_q$ is in $T_q$. So if $\big(\gamma(t),t\in[0,1]\big)$ is a path in $\widetilde{T}_q$ starting and ending at $i\mathbf{1}_q$ then we can deform it by homotopy to $\gamma_s(t)=\gamma(t)+is(s-1)\mathbf{1}_q$, which for $s>0$ is a path inside $T_q$. But $T_q$ as a tube-type domain is simply connected. □

The function $z\mapsto\det(z)$ is a continuous function from $\widetilde{T}_q$ into $\mathbf{C}^*$. From Lemma 3.10, there exists a unique continuous determination of the argument of $\det(z)$ denoted by $\arg\det\colon\widetilde{T}_q\longrightarrow\mathbf{R}$ such that $\arg\det i\mathbf{1}_q=q\frac{\pi}{2}$. If $Y\in\Omega_q$, then $\arg\det iy=q\frac{\pi}{2}$. If $z\in\widetilde{T}_q$ and $g\in\mathrm{GL}(q,\mathbf{C})$, then $\det gzg^*=|\det g|^2\det z$, and $gi\mathbf{1}_qg^*=igg^*\in i\Omega_q$, so that

$$\arg\det gzg^*=\arg\det z\,.$$

This provides a new invariant for the action of $\mathrm{GL}(q,\mathbf{C})$ on $\widetilde{T}_q$.

LEMMA 3.11. *Let* $\Lambda=\Lambda(\lambda_1,\ldots,\lambda_{n_1},n_2,n_3,n_4)$. *Then*

$$\arg\det\Lambda=\arg(\lambda_1+i)+\cdots+\arg(\lambda_{n_1}+i)+n_2\pi+n_4\pi \tag{23}$$

*where* arg *is used for the principal determination of the argument of a non-zero complex number.*

*Proof.* We need to describe a continuous path from $i\mathbf{1}_q$ to $\Lambda$ inside $\widetilde{T}_q$. For clarity of exposition, we describe successively the path for each diagonal block (either a one-dimensional or a two-dimensional submatrix) of $\Lambda$, and compute the contribution of each block to the function $\arg\det$.

For a block of the form $\lambda+i$, with $\lambda\in\mathbf{R}$ we use the path $t\mapsto t\lambda+i$, $0\leq t\leq 1$, and so the contribution of this block is $\arg(\lambda+i)$.

For a block of the form $\begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}$, we use the path

$$t \mapsto \begin{pmatrix} i & t \\ t & i(1-t^2) \end{pmatrix}, \quad 0 \leq t \leq 1 .$$

The corresponding determinant of this $2 \times 2$-block is constant along the path and equal to $-1$. Hence the contribution of this block is $2\frac{\pi}{2} = \pi$.

For a block of the form $1$, we use the path $t \mapsto e^{i\frac{\pi}{2}(1-t)}$, $0 \leq t \leq 1$, and we see that the corresponding contribution is $0$.

For a block of the form $-1$, we use the path $t \mapsto e^{i\frac{\pi}{2}(1+t)}$, $0 \leq t \leq 1$, and we see that the corresponding contribution is $\pi$.

Putting together the contribution of the blocks, we get the result. $\square$

COROLLARY 3.12. *Let $\Lambda$ and $\Lambda'$ be two standard matrices. Assume that their angular matrices coincide and that* $\arg\det\Lambda = \arg\det\Lambda'$. *Then* $\Lambda = \Lambda'$.

*Proof.* In fact we noticed that the equality of angular matrices implies the equality of the parameters except for $n_3 = n_3'$ and $n_4 = n_4'$. But from (23), we see that the equality of the determination of the arguments of the determinants implies $n_4 = n_4'$ (and hence $n_3 = n_3'$). $\square$

Now we can state the conclusion of this section, which is a consequence of Theorem 3.4 and Corollary 3.12.

THEOREM 3.13. *Let $z, z' \in \widetilde{T}_q$, and assume that the angular matrices of $z$ and $z'$ are conjugate, and that* $\arg\det z = \arg\det z'$. *Then $z$ and $z'$ belong to the same orbit under the action of* $\mathrm{GL}(q, \mathbf{C})$.

REMARK. Let $z \in \widetilde{T}_q$. Let $a = {z^*}^{-1} z$. Then

$$\det a = \frac{\det z}{\overline{\det z}} = |\det z|^{-2} (\det z)^2 .$$

So $2 \arg\det z$ is a determination of $\arg(\det a)$. If $z$ and $z'$ are two matrices in $\widetilde{T}_q$ with the same angular matrix, then $\arg\det z$ and $\arg\det z'$ differ by an integral multiple of $\pi$. So the new invariant needed to characterize the orbits under $\mathrm{GL}(q, \mathbf{C})$ has to be regarded as a $\mathbf{Z}$-valued function. In this sense, it is a generalization of the signature.