1. Intersecting chords in convex domains

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The Hilbert metric has found several applications, see [Bi57], [Li95] and [Me95] just to mention a few instances. Typically the idea is to apply the contraction mapping principle to maps which do not increase Hilbert distances (e.g. affine maps).

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1. Intersecting chords in convex domains

From elementary school we know that if c_1, c_2 are two intersecting chords in a circle, then $l_1 l_1' = l_2 l_2'$ where l_1, l_1' and l_2, l_2' denote the respective lengths of the segments into which the two chords are divided. (This follows immediately from the similarity of the associated triangles, see Fig. 1.)

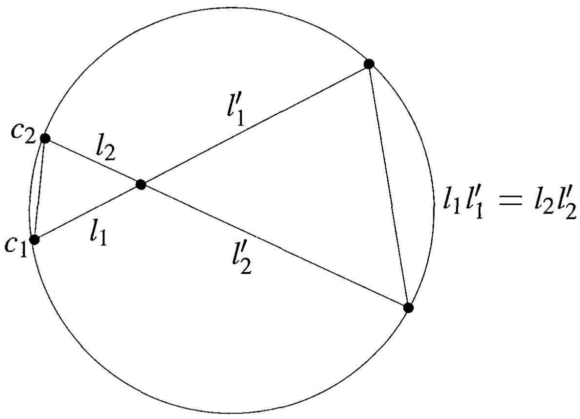

FIGURE 1
Intersecting chords in a circle

A generalization of this fact to any bounded strictly convex domain was given by Beardon in [Be97] by an elegant argument using the Hilbert metric. He proved that if D is such a domain then for each positive δ there is a positive number $M = M(D, \delta)$ such that for any intersecting chords c_1, c_2, each of length at least δ, one has

$$M^{-1} \leq \frac{l_1 l_1'}{l_2 l_2'} \leq M, \tag{1.1}$$

where l_1, l_1', and l_2, l_2' denote the respective lengths of the segments into which the two chords are divided.

We say that a domain satisfies the *intersecting chords property* (ICP) if (1.1) holds for *any* two intersecting chords c_1 and c_2. It is easy to see that ICP may fail for a general strictly convex domain (at a curvature zero point or a 'corner').

We show in this section that ICP holds for domains that satisfy a certain (non-differentiable) curvature condition. Domains with C^2 boundary of nonvanishing curvature are proved to satisfy this condition in Section 3.

1.1 Intersecting line segments and Menger curvature

This subsection clarifies the relation between the curvature of any triple of endpoints and the ratio considered above that two intersecting line segments define.

Three distinct points A, B and C in the plane, not all on a line, lie on a unique circle. Recall that the radius of this circle is

$$R(A,B,C) = \frac{c}{2\sin\gamma}, \tag{1.2}$$

where c is the length of a side of the triangle ABC and γ is the opposite angle. The reciprocal of R is called the *(Menger) curvature* of these three points and is denoted by $K(A,B,C)$.

Now consider two intersecting line segments as in Fig. 2.

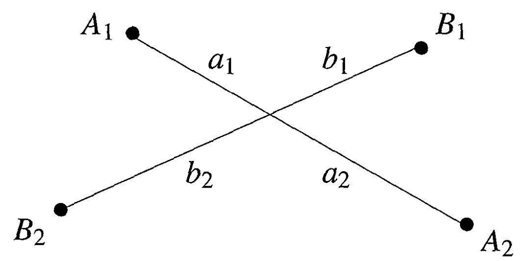

FIGURE 2
Intersecting line segments

PROPOSITION 1.1. *In the above notation, the following equality holds:*

$$\frac{a_1a_2}{b_1b_2} = \frac{K(A_1,B_1,B_2)K(A_2,B_1,B_2)}{K(B_1,A_1,A_2)K(B_2,A_1,A_2)}.$$

Proof. Let α_i be the angle between the line segments A_iB_j and B_1B_2, and let β_i be the angle between B_iA_j and A_1A_2, for $\{i,j\} = \{1,2\}$. By the sine law we have

$$\frac{a_1a_2}{b_1b_2} = \frac{\sin\alpha_1 \sin\alpha_2}{\sin\beta_2 \sin\beta_1} = \frac{2\sin\alpha_1}{|A_1B_1|}\frac{|A_2B_2|}{2\sin\beta_2}\frac{2\sin\alpha_2}{|A_2B_2|}\frac{|A_1B_1|}{2\sin\beta_1}$$
$$= \frac{K(A_1,B_1,B_2)K(A_2,B_1,B_2)}{K(B_1,A_1,A_2)K(B_2,A_1,A_2)}. \quad \square$$

COROLLARY 1.2. *Let D be a bounded convex domain in $\mathbf{R}^n$. Assume that there is a constant $C > 0$ such that*

$$\frac{K(x,y,z)}{K(x',y',z')} \leq C$$

for any two triples of distinct points in ∂D all lying in the same 2-dimensional plane. Then D satisfies the intersecting chords property.

Proof. Any two intersecting chords define a plane and by Proposition 1.1 we have

$$\frac{a_1a_2}{b_1b_2} = \frac{K_{\alpha 1}K_{\alpha 2}}{K_{\beta 1}K_{\beta 2}} \leq C^2. \quad \square$$

REMARK 1.3. In view of this subsection it is clear that ICP implies restrictions on the curvature of the boundary, e.g. there cannot be any points of zero curvature. We were however not able to establish the converse of Corollary 1.2.

1.2 CHORDS LARGER THAN δ

The following proposition provides a different approach to the result in [Be97] mentioned above.

PROPOSITION 1.4. *Let D be a bounded convex domain in $\mathbf{R}^n$. Let δ be such that the length of any line segment contained in ∂D is bounded from above by some $\delta' < \delta$. Then there is a constant $C = C(D,\delta) > 0$ such that*

$$C(D,\delta) \leq K(x,y,z) \leq \frac{2}{\delta}, \tag{1.3}$$

whenever $x,y,z \in \partial D$ and $xy \geq \delta$.

Proof. The angle $\alpha(x,y,v) := \angle_y(xy,v)$ is continuous in $x,y \in \mathbf{R}^n$ and $v \in UT_y(\partial D)$, the unit tangent cone at y. The tangent cone at a boundary point y is the union of all hyperplanes containing y but which are disjoint from D. If $[x,y]$ does not lie in ∂D, then $0 < \alpha(x,y,v) < \pi$. The set

$$S = \{(x,y,v) \in \partial D \times \partial D \times UT_y(\partial D) \,:\, xy \geq \delta\}$$

is compact. Hence there is a constant $\alpha_0 > 0$ such that

$$\alpha_0 \leq \alpha(x, y, v) \leq \pi - \alpha_0 \tag{1.4}$$

for every $(x, y, v) \in S$. By the definition of the tangent cone and compactness there is an $\varepsilon > 0$ such that for any $y, z \in \partial D$, $0 < yz < \varepsilon$ there is an element $v \in UT_y(\partial D)$ for which

$$0 \leq \angle_y(yz, v) \leq \alpha_0/2\,. \tag{1.5}$$

The estimates (1.4) and (1.5) imply the existence of $C > 0$ and the other inequality in (1.3) is trivial. □

As an immediate consequence of Propositions 1.1 and 1.4 we have:

COROLLARY 1.5 (cf. [Be99]). *Let D be a bounded convex domain such that any line segment in ∂D has length less than $\delta' < \delta$. Then the intersecting chords property holds for any two chords each of length greater than δ.*

2. HYPERBOLICITY OF HILBERT'S METRIC

Let (Y, d) be a metric space. Given two points $z, w \in Y$, let

$$(z \mid w)_y = \frac{1}{2}(d(z, y) + d(w, y) - d(z, w))$$

be their *Gromov product* relative to y. We think of y as a fixed base point. The metric space Y is *Gromov hyperbolic* (or *δ-hyperbolic*) if there is a constant $\delta \geq 0$ such that the inequality

$$(x \mid z)_y \geq \min\{(x \mid w)_y, (w \mid z)_y\} - \delta$$

holds for any four points x, y, z, w in Y. As is known, it is enough to show such an inequality for a fixed y (the δ changes by a factor of 2); see [BH99] for a proof of this and we also refer to this book for a general exposition of this important notion of hyperbolicity. By expanding the terms the above inequality is equivalent to

$$d(x, z) + d(y, w) \leq \max\{d(x, y) + d(z, w), d(y, z) + d(x, w)\} + 2\delta\,. \tag{2.1}$$