2. Hyperbolicity of Hilbert's metric

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is compact. Hence there is a constant $\alpha_0 > 0$ such that

$$\alpha_0 \leq \alpha(x,y,v) \leq \pi - \alpha_0 \tag{1.4}$$

for every $(x,y,v) \in S$. By the definition of the tangent cone and compactness there is an $\varepsilon > 0$ such that for any $y,z \in \partial D$, $0 < yz < \varepsilon$ there is an element $v \in UT_y(\partial D)$ for which

$$0 \leq \angle_y(yz, v) \leq \alpha_0/2 . \tag{1.5}$$

The estimates (1.4) and (1.5) imply the existence of $C > 0$ and the other inequality in (1.3) is trivial. □

As an immediate consequence of Propositions 1.1 and 1.4 we have:

COROLLARY 1.5 (cf. [Be99]). *Let D be a bounded convex domain such that any line segment in ∂D has length less than $\delta' < \delta$. Then the intersecting chords property holds for any two chords each of length greater than δ.*

2. HYPERBOLICITY OF HILBERT'S METRIC

Let (Y,d) be a metric space. Given two points $z,w \in Y$, let

$$(z \mid w)_y = \frac{1}{2}(d(z,y) + d(w,y) - d(z,w))$$

be their *Gromov product* relative to y. We think of y as a fixed base point. The metric space Y is *Gromov hyperbolic* (or *δ-hyperbolic*) if there is a constant $\delta \geq 0$ such that the inequality

$$(x \mid z)_y \geq \min\{(x \mid w)_y, (w \mid z)_y\} - \delta$$

holds for any four points x,y,z,w in Y. As is known, it is enough to show such an inequality for a fixed y (the δ changes by a factor of 2); see [BH99] for a proof of this and we also refer to this book for a general exposition of this important notion of hyperbolicity. By expanding the terms the above inequality is equivalent to

$$d(x,z) + d(y,w) \leq \max\{d(x,y) + d(z,w), d(y,z) + d(x,w)\} + 2\delta . \tag{2.1}$$

THEOREM 2.1. *Let D be a bounded convex domain in $\mathbf{R}^n$ satisfying the intersection chords property. Then the metric space (D, h) is Gromov hyperbolic.*

Proof. Suppose that the intersecting chords property holds with a constant M. Let y be a fixed reference point and consider any other three points x, z, w in D. Set $A(u,v) = h(u,v) + h(w,y) - h(u,w) - h(v,y)$ for any two points u, v. By (2.1) we need to show that there is a constant δ independent of x, z, w such that

$$\min\{A(x,z), A(z,x)\} \leq 2\delta \,. \tag{2.2}$$

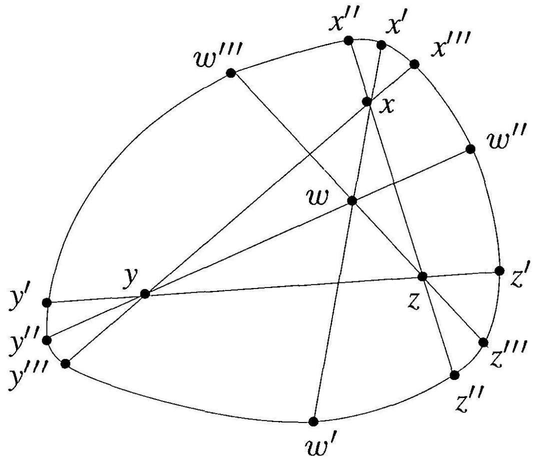

FIGURE 3
Four points

Using the definition of h and the notation in Fig. 3, we have (by rearranging the terms of the product)

$$A(x,z) = \log\left(\frac{xz'' \cdot zx''}{xx'' \cdot zz''} \frac{wy'' \cdot yw''}{ww'' \cdot yy''} \frac{xx' \cdot ww'}{xw' \cdot wx'} \frac{zz' \cdot yy'}{zy' \cdot yz'}\right)$$

$$= \log\left(\frac{xx' \cdot xz''}{xx'' \cdot xw'} \frac{yy' \cdot yw''}{yy'' \cdot yz'} \frac{zz' \cdot zx''}{zz'' \cdot zy'} \frac{ww' \cdot wy''}{ww'' \cdot wx'}\right).$$

Hence, by using $\frac{xx'}{xx''} \leq M\frac{xz''}{xw'}$ and similar inequalities for the other fractions,

$$A(x,z) \leq M' + 2\log\left(\frac{xz''}{xw'} \frac{yw''}{yz'} \frac{zx''}{zy'} \frac{wy''}{wx'}\right).$$

Now, y is fixed and zy', wy'' are bounded from above and below respectively, so that

$$A(x,z) \leq M'' + 2\log\left(\frac{xz'' \cdot zx''}{xw' \cdot wx'}\right).$$

So (2.2) is equivalent to the boundedness of

$$\min\left\{\frac{xz'' \cdot zx''}{xw' \cdot wx'}, \frac{zx'' \cdot xz''}{zw''' \cdot wz'''}\right\}$$

from above. By symmetry we may assume without loss of generality that $zz'' \leq xx''$. Now we have two cases:

CASE 1: *$xw \geq xx''$ or $zw \geq xx''$.*

If $xw \geq zw$ (so in particular $xw \geq xx''$), then

$$\begin{aligned}\frac{xz'' \cdot zx''}{xw' \cdot wx'} &\leq \frac{(xz + zz'')(zx + xx'')}{(xw)^2} \\ &\leq \frac{(xw + wz + zz'')(zw + wx + xx'')}{(xw)^2} \leq \frac{(3xw)^2}{(xw)^2} \leq 9 .\end{aligned}$$

When $zw \geq xw$, we estimate the other fraction instead (obtained by interchanging x and z) in the same way.

CASE 2: *$xw \leq xx''$ and $zw \leq xx''$.*

Considering chords at x we have

$$\frac{xz'' \cdot zx''}{xw' \cdot wx'} \leq M \frac{xx' \cdot zx''}{xx'' \cdot wx'} \leq M \frac{xx'}{wx'} \frac{(xw + wz + xx'')}{xx''} \leq 3M$$

since $xx'' \cdot xz'' \leq M(xx' \cdot xw')$. □

REMARK 2.2. Since the n-dimensional ball B^n obviously satisfies the assumption in Corollary 1.2 with $C = 1$, Theorem 2.1 contains the standard fact that (B^n, h), which is Klein's model of the n-dimensional hyperbolic space, is Gromov hyperbolic.

REMARK 2.3. The above proof does not appeal to compactness and therefore goes through in infinite dimensions provided that y lies at positive distance from the boundary. In particular, it proves that the unit ball in a Hilbert space with the Hilbert metric, which is the infinite dimensional hyperbolic space, is Gromov hyperbolic. Note however that ICP is not affinely invariant in infinite dimensions. (Kaimanovich brought this remark to our attention.)