2. Tetrahedra

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If X is a divisor in M the bundle thus defined is equal to $\mathcal{O}_M(X)|_D$. In particular, if X is a central fibre in a semistable degeneration $\mathcal{X} \to S$, then $\mathcal{O}_{\mathcal{X}}(X) \equiv \mathcal{O}_{\mathcal{X}}$ so $\mathcal{O}_D(X) = \mathcal{O}_D$. This gives a necessary condition for being a central fibre.

1.8. DEFINITION. The normal crossings surface X is *d-semistable* if $\mathcal{O}_D(X) = \mathcal{O}_D$.

A consequence is the triple point formula: let $D_{ij} = X_i \cap X_j$ and denote by $(D_{ij})^2_{X_i}$ the self intersection of D_{ij} on X_i and by T_{ij} the number of triple points on D_{ij}. Then (cf. [P, Cor. 2.4.2])

$$(D_{ij})^2_{X_i} + (D_{ij})^2_{X_j} + T_{ij} = 0\,.$$

1.9. DEFINITION. A compact normal crossings surface is a *d-semistable $K3$-surface of type* III if X is d-semistable, $\omega_X = \mathcal{O}_X$ and each X_i is rational, the double curves $D_i \subset X_i$ are cycles of rational curves and the dual graph triangulates S^2. If the conclusions of the Minus One Theorem hold, that every component of the double curve has self intersection -1 on either component of X on which it lies, the surface X is said to be in *(-1)-form.*

2. TETRAHEDRA

2.1. To realise a tetrahedron we start out with four general planes in 3-space. They do not form a d-semistable $K3$-surface, but the dual graph is a tetrahedron. To write down a degeneration with this special fibre we just take the pencil spanned by $T = x_0x_1x_2x_3$ and a smooth quartic. The symmetry group of the tetrahedron (including reflections) acts if we only take S_4-invariant quartics:

$$Q = a\sigma_1^4 + b\sigma_1^2\sigma_2 + c\sigma_2^2 + d\sigma_1\sigma_3\,,$$

where the σ_i are the elementary symmetric functions in the four variables x_i and a, b, c and d are constants.

To obtain a family $f\colon \mathcal{X} \to S$ one has to blow up the base locus of the pencil. This can be done in several ways. Blowing up $T = Q = 0$ gives a total space which is singular, with in general 24 ordinary double points coming from the 24 intersection points of Q with the double curve of the tetrahedron T. Arguably, this is the nicest model, and the best one can hope for in view of the theory of minimal models of 3-folds. A smooth model is obtained by a suitable small resolution of the 24 singularities. The quartic intersects each edge of the tetrahedron in four points. To get the (-1)-form two of them have to be blown up in one face and the other two in the other face. The central fibre then consists of four Del Pezzo surfaces of degree 3.

Alternatively one can blow up the irreducible components of $T = Q = 0$ one at a time. The advantage is that one has a projective model. However it is not in (-1)-form and furthermore the symmetry is not preserved. To achieve (-1)-form we have to apply modifications of type I; here we might lose projectivity.

2.2. THE TETRAHEDRON OF DEGREE 12. We glue together four Del Pezzo surfaces of degree 3. Take coordinates $x_1, \ldots, x_4, y_1, \ldots, y_4$ on $\mathbf{P}^7$. Let $\{i,j,k,l\} = \{1,2,3,4\}$. The Del Pezzo surface X_i lies in $y_i = x_j = x_k = x_l = 0$ and has an equation of the form

$$y_j y_k y_l - x_i F_i(x_i, y_j, y_k, y_l) = 0,$$

where F_i is a quadratic form; more specifically,

$$F_i = \sum_{\alpha\neq i} f_i^{\alpha\alpha} y_\alpha^2 + \sum_{\alpha,\beta\neq i} f_i^{\alpha\beta} y_\alpha y_\beta + \sum_{\alpha\neq i} g_i^\alpha x_i y_\alpha + h_i x_i^2 .$$

The condition that X_i be nonsingular in the vertices of the triangle $x_i = y_j y_k y_l = 0$ is that the coefficients $f_i^{\alpha\alpha}$ in F_i do not vanish.

The ideal of the tetrahedron $X = \bigcup_i X_i$ has 14 generators, the 4 cubic Del Pezzo equations $y_j y_k y_l - x_i F_i$ and 10 quadratic monomials: the six products $x_i x_j$ and the four products $x_i y_i$. The relations among them are:

$$\begin{aligned}
&(x_i x_j)\, x_k - (x_i x_k)\, x_j \\
&(x_i y_i)\, x_j - (x_i x_j)\, y_i \\
&(y_j y_k y_l - x_i F_i)\, x_l - (x_l y_l)\, y_j y_k + (x_i x_l)\, F_i \\
&(y_i y_j y_k - x_l F_l)\, y_l - (y_j y_k y_l - x_i F_i)\, y_i - (x_l y_l)\, F_l + (x_i y_i)\, F_i .
\end{aligned} \tag{2.1}$$

2.3. PROPOSITION. *The tetrahedron X is d-semistable if and only if the four equations*

$$f_k^{jj} f_l^{kk} f_j^{ll} - f_j^{kk} f_k^{ll} f_l^{jj} = 0$$

are satisfied.

Proof. We look at the chart $y_4 = 1$. Then $x_4 = 0$ and we have the equations $x_i x_j$, $x_i y_i$, $y_1 y_2 y_3$ and $y_i y_j - x_k F_k$. In all points near the origin $y_1 y_2 y_3 \mapsto 1$ is generator of the infinitesimal normal bundle $\mathcal{O}_D(X)$. We now look on the y_2-axis. The equation $y_2y_1 - x_3F_3$ shows that the section $y_1y_2y_3 \mapsto 1$ has a pole in the zeroes of F_3 restricted to the y_2-axis, and likewise in the zeroes of F_1 (using the equation $y_2y_3 - x_1F_1$).

This shows that the expression, given in homogeneous coordinates by

$$\frac{(f_1^{22}y_2^2 + f_1^{24}y_2y_4 + f_1^{44}y_4^2)(f_3^{22}y_2^2 + f_3^{24}y_2y_4 + f_3^{44}y_4^2)}{y_1y_2y_3y_4},$$

represents a non vanishing holomorphic section of $\mathcal{O}_D(X)$ on the whole line $y_1 = y_3$. Similar expression can be found for the other edges of the tetrahedron. We get a global section if and only if we can find a quaternary form f of degree 4 which restricts to a multiple of the above denominator for each line.

For each face we find from the following lemma the condition that the 12 points in the corresponding hyperplane be cut out by a quartic. We obtain four equations, which in fact are not independent: under our assumption that all f_{jj}^i are different from 0 one can derive one equation from the remaining three. They give necessary and sufficient conditions for the existence of the quaternary quartic. □

2.4. Lemma. *Consider $3n$ points $P_{i,\alpha}$ with $P_{i,1}$, ..., $P_{i,n}$ smooth points of the triangle $x_1x_2x_3 = 0$, lying on the side $x_i = 0$ and given by the binary form $B_i(x_j, x_k) = \sum_{m=0}^{n} b_{im}\, x_j^m x_k^{n-m}$, where (i,j,k) is a cyclic permutation of* $(1,2,3)$. *These points are cut out by a ternary form of degree n if and only if*

$$b_{10}b_{20}b_{30} = b_{1n}b_{2n}b_{3n}\,.$$

Proof. Suppose $A(x_1,x_2,x_3) = \sum_{l+m+p=n} a_{lmp}\, x_1^l x_2^m x_3^p$ cuts out the points. Then $A(0,x_2,x_3)$ is proportional to $B_1(x_2,x_3)$, so $(a_{0n0} : a_{00n}) = (b_{10} : b_{1n})$. Likewise we have that $(a_{00n} : a_{n00}) = (b_{20} : b_{2n})$ and that $(a_{n00} : a_{0n0}) = (b_{30} : b_{3n})$. Multiplying these ratios gives the condition.

Conversely, to find A we may suppose that $b_{10} = b_{3n} \doteq 1 = a_{0n0}$ (as no point lies at one of the vertices). We put $A(0,x_2,x_3) = B_1(x_2,x_3)$, $A(x_1,x_2,0) = B_3(x_1,x_2)$. We also can take $b_{20} = a_{00n}$. As $b_{1n} = a_{00n}$ the condition gives now $b_{2n} = b_{30} = a_{n00}$ and we can set $A(x_1,0,x_3) = B_2(x_1,x_3)$. The remaining monomials in A are divisible by $x_1x_2x_3$ and do not matter. □

2.5. Remark. It is not surprising that only the extremal coefficients b_{i0}, b_{in} are involved, as they depend only on the product of the coordinates of the points. Ignoring the other coefficients we rename: $b_{i0} =: b_{jk}$, $b_{in} =: b_{kj}$. The condition becomes $b_{jk}b_{ki}b_{ij} = b_{ji}b_{ik}b_{kj}$, which is the form used in the proposition above.

2.6. INFINITESIMAL DEFORMATIONS. We compute embedded deformations modulo coordinate transformations. To this end we look at the equations as defining the affine cone $C(X)$ over X. We follow the standard procedure (see e.g. [S1]): given equations f_i, satisfying relations $\sum f_i r_{ij} = 0$, we have to lift the equations to $F_i = f_i + \varepsilon f_i'$ and the relations to $R_{ij} = r_{ij} + r_{ij}'$, satisfying $\sum F_i R_{ij} \equiv 0 \pmod{\varepsilon^2}$. This means that we have to find f_i' such that $\sum f_i' r_{ij}$ lies in the ideal generated by the f_i. Using undetermined coefficients this is a finite dimensional problem for each degree. The deformations of $C(X)$ in degree 0 give embedded deformations of X in $\mathbf{P}^7$, while those in degree < 0 have an interpretation in terms of extensions of X: they tell us of which varieties X is a hyperplane section. Our main interest lies in the degree 0 deformations, but as preparation we first compute those of negative degrees.

2.7. PROPOSITION. *The dimension of* $T^1_{C(X)}(-2)$ *equals 4 and we have* $\dim T^1_{C(X)}(-1) = 16$. *In case X is d-semistable* $\dim T^1_{C(X)}(0) = 22$, *otherwise it is* 21.

Proof. *Degree* -2: we perturb the quadratic equations with constants and the cubic equations with linear terms. Write $x_i x_j + a_{ij}$. The first type of the relations (2.1) then gives $a_{ij} x_k - a_{ik} x_j = 0 \in \mathcal{O}_{C(X)}$, so $a_{ik} = 0$. Also the equation $x_i y_i$ are not perturbed. Consider $y_j y_k y_l - x_i F_i + \sum a_i^\alpha x_\alpha + \sum b_i^\alpha y_\alpha$. The third relation gives $a_i^j x_j^2 + \sum_{\alpha \neq j} b_i^\alpha x_j y_\alpha = 0$ so we conclude that all coefficients vanish, except the a_i^i, which we may choose arbitrary. The last type of relation is then also satisfied.

Degree -1: consider the perturbations

$$x_i x_j + \sum a_{ij}^\alpha x_\alpha + \sum b_{ij}^\alpha y_\alpha .$$

In the local ring we obtain the equation

$$a_{ij}^k x_k^2 + \sum_{\alpha \neq k} b_{ij}^\alpha x_k y_\alpha - a_{ik}^k x_j^2 + \sum_{\alpha \neq j} b_{ik}^\alpha x_j y_\alpha = 0 ,$$

from which we get $b_{ij}^\alpha = 0$, $a_{ij}^k = a_{ij}^l = 0$. We now put

$$x_i y_i + \sum a_{ii}^\alpha x_\alpha + \sum b_{ii}^\alpha y_\alpha .$$

We find

$$a_{ii}^j x_j^2 + \sum_{\alpha \neq j} b_{ii}^\alpha x_j y_\alpha - a_{ij}^j x_j y_i = 0 .$$

We conclude $a_{ii}^j = 0$, $b_{ii}^j = 0$ for all $j \neq i$ and finally $a_{ij}^j = b_{ii}^i$. In particular a_{ij}^j is independent of j. We can use the coordinate transformation $x_i \mapsto x_i - b_{ii}^i$

to get rid of the a_{ij}^j-term. So the equations x_ix_j are not perturbed at all. This means that x_iy_j is only perturbed with the term $a_{ii}^ix_i$, which can be made to vanish by coordinate transformations in the y_i-variables. As above we find that the only allowable perturbations of the cubic equation $y_jy_ky_l - x_iF_i$ are those divisible by x_i. As we have used all coordinate transformations, all monomials x_i^2, x_iy_j can occur. This makes the dimension of $T^1(-1)$ into 4×4.

Degree 0: we proceed in the same way by first considering the perturbations

$$x_ix_j + \sum a_{ij}^\alpha x_\alpha^2 + \sum b_{ij}^{\alpha\beta} x_\alpha y_\beta + \sum c_{ij}^\alpha y_\alpha^2 + \sum d_{ij}^{\alpha\beta} y_\alpha y_\beta \,.$$

Multiplied with x_k this gives the following terms in the local ring:

$$a_{ij}^k x_k^3 + \sum_{\beta\neq k} b_{ij}^{k\beta} x_k^2 y_\beta + \sum_{\alpha\neq k} c_{ij}^\alpha x_k y_\alpha^2 + \sum_{\alpha,\beta\neq k} d_{ij}^{\alpha\beta} x_k y_\alpha y_\beta \,.$$

We conclude that all coefficients occurring here vanish. In particular $a_{ij}^k = 0$. Using the coordinate transformations $x_j \mapsto x_j - a_{ij}^i x_i$ we may suppose that all a_{ij}^α vanish. We are left with

$$x_ix_j + \sum b_{ij}^{i\beta} x_i y_\beta + \sum b_{ij}^{j\beta} x_j y_\beta + d_{ij}^{kl} y_k y_l \,.$$

With the perturbations

$$x_iy_i + \sum_{\alpha\neq i} a_{ii}^\alpha x_\alpha^2 + \sum_{\alpha\neq i} b_{ii}^{\alpha\beta} x_\alpha y_\beta + \sum_{\alpha\neq i} c_{ii}^\alpha y_\alpha^2 + \sum_{\alpha,\beta\neq i} d_{ii}^{\alpha\beta} y_\alpha y_\beta \,,$$

where we used coordinate transformations $x_i \mapsto x_i - c_{ii}^i y_i$, $x_i \mapsto x_i - d_{ii}^{ij} y_j$, $y_i \mapsto y_i - a_{ii}^i x_i$ and $y_i \mapsto y_i - b_{ii}^{ij} y_j$ to remove some coefficients, we now get (using the jth Del Pezzo equation)

$$a_{ii}^j x_j^3 + \sum b_{ii}^{j\beta} x_j^2 y_\beta + \sum_{\alpha\neq i} c_{ii}^\alpha x_j y_\alpha^2 + \sum_{\alpha,\beta\neq i} d_{ii}^{\alpha\beta} x_j y_\alpha y_\beta - \sum b_{ij}^{j\beta} x_j y_i y_\beta - d_{ij}^{kl} x_j F_j = 0 \,.$$

Using the explicit expression for F_j we obtain the equations

$$\begin{aligned} a_{ii}^j &= d_{ij}^{kl} h_j \,, & b_{ii}^{j\beta} &= d_{ij}^{kl} g_j^\beta \,, & c_{ii}^\alpha &= d_{ij}^{kl} f_j^{\alpha\alpha} \,, \\ -b_{ij}^{ji} &= d_{ij}^{kl} f_j^{ii} \,, & d_{ii}^{kl} &= d_{ij}^{kl} f_j^{kl} \,, & -b_{ij}^{j\beta} &= d_{ij}^{kl} f_j^{i\beta} \,. \end{aligned}$$

We can determine all coefficients, but because c_{ii}^α does not depend on j, we get two equations for it:

$$d_{ij}^{kl} f_j^{ll} = c_{ii}^l = d_{ik}^{jl} f_k^{ll} \,.$$

We view these as one linear equation for the unknowns d_{ij}^{kl}. The coefficient matrix of the resulting linear system is

$$\begin{pmatrix} 0 & f_k^{jj} & -f_l^{jj} & 0 & 0 & 0 \\ -f_j^{kk} & 0 & f_l^{kk} & 0 & 0 & 0 \\ f_j^{ll} & -f_k^{ll} & 0 & 0 & 0 & 0 \\ f_i^{kk} & 0 & 0 & 0 & -f_l^{kk} & 0 \\ -f_i^{ll} & 0 & 0 & f_k^{ll} & 0 & 0 \\ 0 & 0 & 0 & -f_k^{ii} & f_l^{ii} & 0 \\ 0 & f_i^{ll} & 0 & -f_j^{ll} & 0 & 0 \\ 0 & -f_i^{jj} & 0 & 0 & 0 & f_l^{jj} \\ 0 & 0 & 0 & f_j^{ii} & 0 & -f_l^{ii} \\ 0 & 0 & f_i^{jj} & 0 & 0 & -f_k^{jj} \\ 0 & 0 & -f_i^{kk} & 0 & f_j^{kk} & 0 \\ 0 & 0 & 0 & 0 & -f_j^{ii} & f_k^{ii} \end{pmatrix}.$$

It has a nontrivial solution if all 6×6 minors vanish. Among those are

$$f_j^{kk} f_k^{ll} f_l^{jj} (f_k^{jj} f_l^{kk} f_j^{ll} - f_j^{kk} f_k^{ll} f_l^{jj})$$

from which we obtain that the square of

$$f_k^{jj} f_l^{kk} f_j^{ll} - f_j^{kk} f_k^{ll} f_l^{jj}$$

lies in the ideal of the minors. This is one of the four conditions for d-semistability. There are three more equations

$$f_k^{jj} f_j^{ll} f_l^{ii} f_i^{kk} - f_j^{kk} f_l^{jj} f_i^{ll} f_k^{ii}$$

in the reduction of the ideal of minors, which do not give new conditions if the $f_i^{jj} \neq 0$, as

$$\begin{aligned} f_l^{kk}(f_k^{jj} f_j^{ll} f_l^{ii} f_i^{kk} - f_j^{kk} f_l^{jj} f_i^{ll} f_k^{ii}) \\ = f_l^{ii} f_i^{kk} (f_k^{jj} f_l^{kk} f_j^{ll} - f_j^{kk} f_k^{ll} f_l^{jj}) + f_j^{kk} f_l^{jj} (f_l^{ii} f_i^{kk} f_k^{ll} - f_i^{ll} f_k^{ii} f_l^{kk}). \end{aligned}$$

Under the d-semistability conditions the rank of the matrix is 5, and we obtain one infinitesimal deformation, where the quadratic equations are perturbed. Furthermore one has the perturbations of the cubic equations alone, which as before have to be divisible by x_i. We have already used 44 coordinate transformations. The coefficient of $x_i y_\alpha y_\beta$ can be made to vanish with a transformation of the type $y_\gamma \mapsto y_\gamma - \varepsilon x_i$. So we have 28 coefficients left and the diagonal coordinate transformations, giving dimension 21. □

The computations in negative degree show that the tetrahedron X is only a hyperplane section of threefolds with two-dimensional singular locus, obtained by gluing together four cubic threefolds.

2.8. We want to describe an explicit deformation in the d-semistable case. We use the coordinate transformation $x_i \mapsto (f_4^{ii}/f_i^{44})x_i$, $i = 1, 2, 3$, which gives $f_i^{jj}f_4^{ii}/f_i^{44}$ as coefficient of y_j^2 in the new F_i. The d-semistability conditions yield that the new coefficients satisfy $f_i^{jj} = f_j^{ii}$. We will denote them by f_{ij}. A solution to the linear equations above is then $d_{ij} = df_{kl}$, with d a new deformation variable. Furthermore we use coordinate transformations to remove the $y_\alpha y_\beta$ terms from the F_i.

We set $H_i = g_i^j y_j + g_i^k y_k + g_i^l y_l + h_i x_i$. With this notation we get the following infinitesimal deformation:

$$x_i x_j + df_{kl} y_k y_l - df_{ij} f_{kl}(x_i y_j + x_j y_i)\,,$$
$$x_i y_i + d(f_{kl}\, x_j H_j + f_{jl}\, x_k H_k + f_{jk}\, x_l H_l + f_{jk} f_{jl} y_j^2 + f_{kl} f_{kj} y_k^2 + f_{lj} f_{lk} y_l^2)\,,$$
$$\begin{aligned} y_j y_k y_l - x_i(f_{ij} y_j^2 + f_{ik} y_k^2 + f_{il} y_l^2) - x_i^2 H_i \\ + dy_i\big((f_{ik} f_{jl} + f_{il} f_{jk}) f_{ij} y_j^2 + (f_{ij} f_{kl} + f_{il} f_{kj}) f_{ik} y_k^2 + (f_{ij} f_{lk} + f_{ik} f_{lj}) f_{il} y_l^2\big) \\ + df_{ij} f_{ik} f_{il} y_i^3 + dy_i(f_{ik} f_{il}\, x_j H_j + f_{ij} f_{il}\, x_k H_k + f_{ij} f_{ik}\, x_l H_l)\,. \end{aligned}$$

If we try to lift to higher order complicated formulas arise, and it is not clear whether the computation is finite. It does stop if we restrict ourselves to the case of tetrahedral symmetry. Then f_{ij} does not depend on (i,j), and we call the common value f; likewise g is the value of all g_i^j, and h that of the h_i. We retain the notation $H_i = g(y_j + y_k + y_l) + hx_i$. By a coordinate transformation $x_i \mapsto x_i + df^2 y_i$ we simplify the expression for the first 6 equations. We write t for the deformation parameter.

2.9. PROPOSITION. *The following set of equations defines a degeneration of* $K3$*-surfaces with special fibre a tetrahedron of degree* 12 :

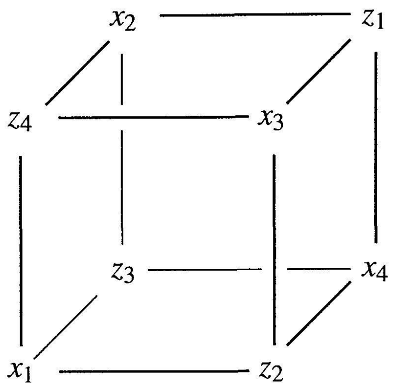
$$x_i x_j + tf(y_k - tfH_k)(y_l - tfH_l)\,,$$
$$x_i y_i + tf(x_j H_j + x_k H_k + x_l H_l) + tf^2(y_i^2 + y_j^2 + y_k^2 + y_l^2)\,,$$
$$\begin{aligned} y_j y_k y_l - f x_i(y_i^2 + y_j^2 + y_k^2 + y_l^2) - x_i^2 H_i \\ - t^2 f^2(y_j H_k H_l + y_k H_j H_l + y_l H_j H_k) + 2t^3 f^3 H_j H_k H_l\,. \end{aligned}$$

The general fibre is a smooth $K3$*-surface lying on* $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

Proof. We suppress the computation. For $t \neq 0$ the cubic equations lie in the ideal of the quadrics. We can write three independent equations

$$x_i(y_i - tfH_i) - x_j(y_j - tfH_j)$$

which together with the first six define $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$: each square in the following picture gives an equation, where we put $z_i := \sqrt{-tf}\,(y_i - tfH_i)$.



We obtain the $K3$ by taking the complete intersection with the quadric $\sum_i(x_iy_i + 3tfx_iH_i + 4tf^2y_i^2)$. □

2.10. THE RELATION WITH QUARTICS. We can construct a degree 12 tetrahedron from four planes in $\mathbf{P}^3$ by first blowing up each plane in 6 points and then gluing them back together. Therefore we first describe the blow up in a way adapted to our situation.

Let two points lie on each side of the coordinate triangle in $\mathbf{P}^2$ with coordinates $(z_1 : z_2 : z_3)$. We describe them by $z_k = z_i^2 + a_{ij}z_iz_j + b_{ij}z_j^2 = 0$, where (ijk) is a cyclic permutation of (123) (this means that we choose an orientation on the triangle). As cubics through the six points we take the coordinate triangle and three cubics, each consisting of a side and a quadric passing through the remaining four points. More precisely, we take

$$x_0 = z_1z_2z_3\,,$$
$$y_i = z_i(z_k^2 - a_{ki}z_kz_i + b_{ki}(z_i^2 - a_{ij}z_iz_j + b_{ij}z_j^2))\,.$$

One computes the relations

$$z_ky_j - b_{ij}z_jy_k - ((1 - b_{ij}b_{jk}b_{ki})\,z_i - a_{ij}z_j + b_{ij}b_{jk}a_{ki}z_k)x_0 = 0\,.$$

By the Hilbert-Burch theorem the maximal minors of the relation matrix

$$\begin{pmatrix} 0 & z_3 & -b_{12}z_2 & -(1 - b_{12}b_{23}b_{31})\,z_1 + a_{12}z_2 - b_{12}b_{23}a_{31}z_3 \\ -b_{23}z_3 & 0 & z_3 & -(1 - b_{12}b_{23}b_{31})\,z_2 + a_{23}z_3 - b_{23}b_{31}a_{12}z_1 \\ z_2 & -b_{31}z_1 & 0 & -(1 - b_{12}b_{23}b_{31})\,z_3 + a_{31}z_1 - b_{31}b_{12}a_{23}z_2 \end{pmatrix}$$

give the cubics, up to a common factor $1 - b_{12}b_{23}b_{31}$. By Lemma 2.4 this factor vanishes exactly when the 6 points lie on a conic.

Viewing the relations as holding between the z_i gives the coefficient matrix

$$\begin{pmatrix} (1-b_{12}b_{23}b_{31})x_0 & b_{12}y_3 - a_{12}x_0 & b_{12}b_{23}a_{31}x_0 - y_2 \\ b_{23}b_{31}a_{12}x_0 - y_3 & (1-b_{12}b_{23}b_{31})x_0 & y_1b_{23} - a_{23}x_0 \\ b_{31}y_2 - a_{31}x_0 & b_{12}b_{31}a_{23}x_0 - y_1 & (1-b_{12}b_{23}b_{31})x_0 \end{pmatrix}.$$

Its determinant is the equation of the surface. After dividing by $b_{12}b_{23}b_{31}-1$ it equals

$$\begin{aligned} &y_1y_2y_3 - x_0(b_{12}y_3^2 + b_{23}y_1^2 + b_{31}y_2^2) \\ &- x_0^2\big(a_{12}a_{23}b_{31}y_2 + a_{23}a_{31}b_{12}y_3 + a_{31}a_{12}b_{23}y_1 - (a_{12}y_3 + a_{23}y_1 + a_{31}y_2)(1 + b_{12}b_{23}b_{31})\big) \\ &- x_0^3\big((1-b_{12}b_{23}b_{31})^2 + (1+b_{12}b_{23}b_{31})a_{12}a_{23}a_{31} + b_{12}b_{23}a_{31}^2 + b_{23}b_{31}a_{12}^2 + b_{12}b_{31}a_{23}^2\big)\,. \end{aligned}$$

This last formula also works if the 6 points lie on a conic, but then it is easier to take the y_i as product of a side and the conic through the 6 points; this means adding a multiple of x_0 to each y_i. The equation then becomes $y_1y_2y_3 - x_0Q(y)$ with $Q(z)$ the conic.

Now we apply this to our tetrahedron. We choose an orientation and orient the faces with the induced orientation. We get variables x_i and y_i. For the face i we take $x_i = z_jz_kz_l$ as before, but we multiply y_j by a factor λ_{ij} to be determined later. So we set $y_j = \lambda_{ij}z_j(z_l^2 + \dots)$. We look at the line $z_3 = z_0 = 0$, with coordinates $(z_1 : z_2)$. Via the coordinates of face 0 we get the embedding $(y_1 : y_2) = (\lambda_{01}b_{31}z_1 : \lambda_{02}z_2)$ whereas face 3 gives $(y_1 : y_2) = (\lambda_{31}z_1 : \lambda_{32}b_{02}z_2)$. The condition that the Del Pezzo surfaces are glued in the same way as the planes yields the equations $\lambda_{01}\lambda_{32}b_{31}b_{02} = \lambda_{02}\lambda_{31}$. By even permutations of (0123) we get in total six equations. They are solvable if and only if

$$b_{01}b_{10}b_{02}b_{20}b_{03}b_{30}b_{12}b_{21}b_{13}b_{31}b_{23}b_{32} = 1\,,$$

a condition obtained by multiplying the six equations.

The d-semistability conditions $f_k^{jj}f_l^{kk}f_j^{ll} - f_j^{kk}f_k^{ll}f_l^{jj} = 0$ give for the cubic above, using $(jkl) = (123)$:

$$\frac{b_{30}}{\lambda_{21}^2}\frac{b_{10}}{\lambda_{32}^2}\frac{b_{20}}{\lambda_{13}^2} = \frac{b_{03}}{\lambda_{12}^2}\frac{b_{01}}{\lambda_{23}^2}\frac{b_{02}}{\lambda_{31}^2}\,.$$

Using the fact that the λ_{ij} satisfy the equations $\lambda_{32}b_{02}/\lambda_{31} = \lambda_{02}/(\lambda_{01}b_{31})$ we see that this condition is equivalent to $b_{21}^2b_{32}^2b_{13}^2 = b_{12}^2b_{23}^2b_{31}^2$, which is one of the conditions that the 24 points be cut out by a quartic.

We can ask which choices of 24 points give our symmetric tetrahedron. The condition $\prod b_{ij} = 1$ limits the possibilities. In particular, if all $b_{ij} = 1$, the six points in each face lie on a conic, giving a singular tetrahedron. If we take the quartic $Q = (a\sigma_1^2 + b\sigma_2)^2$ then each element of the pencil has 12 singular points. We can blow them up and blow down the six conics in the faces by embedding the pencil in $\mathbf{P}^7 \times \mathbf{P}^1$ with the linear system of cubics in $\mathbf{P}^3$ with as base points the 12 singular points. We set

$$x_i = z_j z_k z_l \,,$$
$$y_i = z_i(a\sigma_1^2 + b\sigma_2)\,.$$

We obtain a symmetric tetrahedron with $g = h = 0$.

We get nonsingular Del Pezzo surfaces by taking all $b_{ij} = -1$, and $a_{ij} = a$. Then $f = -1$, $g = -a^2$ and $h = a^2 + 4$. The points on the side of the tetrahedron are given by

$$(z_i^2 + az_iz_j - z_j^2)(-z_i^2 + az_iz_j + z_j^2) = (-z_i^4 + (2+a^2)\,z_i^2 z_j^2 - z_j^4)\,.$$

In particular, we obtain different smoothings of the same tetrahedron, those embedded in $\mathbf{P}^7$ and others where the general fibre is embeddable in $\mathbf{P}^3$. They belong to different 19-dimensional hypersurfaces in the 20-dimensional subspace of the versal deformation whose general fibre is a smooth $K3$-surface.

3. Deformation theory

3.1. Let $X = \bigcup X_i$ be a normal crossings surface with normalisation $\widetilde{X} = \coprod X_i$. The components of the double locus D are $D_{ij} = X_i \cap X_j$. The divisor $D_i := \bigcup_j D_{ij}$ is a normal crossings divisor in X_i. We set $\widetilde{D} = \coprod D_i$.

As X is locally a hypersurface in a 3-fold M, its cotangent cohomology sheaves $\mathcal{T}_X^i$ vanish for $i \geq 2$ and

$$0 \longrightarrow \mathcal{T}_X^0 \longrightarrow \Theta_M|_X \longrightarrow N_{X/M} \longrightarrow \mathcal{T}_X^1 \longrightarrow 0\,.$$

There is a canonical isomorphism $\mathcal{T}_X^1 \cong \mathcal{O}_D(X)$ and in particular, if X is d-semistable, then $\mathcal{T}_X^1 \cong \mathcal{O}_D$ [F2, Prop. 2.3].

3.2. Lemma. *There is an exact sequence*

$$0 \longrightarrow \mathcal{T}_X^0 \longrightarrow n_*\Theta_{\widetilde{X}}(\log \widetilde{D}) \longrightarrow \mathcal{T}_D^0 \longrightarrow 0\,.$$

Proof. This is a local computation. The sheaf $\Theta_M(\log X)$ of vector fields on M which preserve $z_1z_2z_3 = 0$ is generated by the $z_i\frac{\partial}{\partial z_i}$. Restricted to a component X_i: $z_i = 0$ we get sections of $\Theta_{X_i}(\log D_i)$. The restrictions to different components satisfy the obvious compatibility condition. □