Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 48 (2002)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SEMISTABLE K3-SURFACES WITH ICOSAHEDRAL SYMMETRY
Autor: Stevens, Jan
Kapitel: 3. DEFORMATION THEORY
DOI: https://doi.org/10.5169/seals-66069

points. We can blow them up and blow down the six conics in the faces by embedding the pencil in $\mathbf{P}^7 \times \mathbf{P}^1$ with the linear system of cubics in $\mathbf{P}^3$ with as base points the 12 singular points. We set

$$x_i = z_j z_k z_l\,,$$
$$y_i = z_i(a\sigma_1^2 + b\sigma_2)\,.$$

We obtain a symmetric tetrahedron with $g = h = 0$.

We get nonsingular Del Pezzo surfaces by taking all $b_{ij} = -1$, and $a_{ij} = a$. Then $f = -1$, $g = -a^2$ and $h = a^2 + 4$. The points on the side of the tetrahedron are given by

$$(z_i^2 + az_iz_j - z_j^2)(-z_i^2 + az_iz_j + z_j^2) = (-z_i^4 + (2 + a^2)\,z_i^2z_j^2 - z_j^4)\,.$$

In particular, we obtain different smoothings of the same tetrahedron, those embedded in $\mathbf{P}^7$ and others where the general fibre is embeddable in $\mathbf{P}^3$. They belong to different 19-dimensional hypersurfaces in the 20-dimensional subspace of the versal deformation whose general fibre is a smooth $K3$-surface.

3. Deformation theory

3.1. Let $X = \bigcup X_i$ be a normal crossings surface with normalisation $\widetilde{X} = \coprod X_i$. The components of the double locus D are $D_{ij} = X_i \cap X_j$. The divisor $D_i := \bigcup_j D_{ij}$ is a normal crossings divisor in X_i. We set $\bar{D} = \coprod D_i$.

As X is locally a hypersurface in a 3-fold M, its cotangent cohomology sheaves $\mathcal{T}_X^i$ vanish for $i \geq 2$ and

$$0 \longrightarrow \mathcal{T}_X^0 \longrightarrow \Theta_M|_X \longrightarrow N_{X/M} \longrightarrow \mathcal{T}_X^1 \longrightarrow 0\,.$$

There is a canonical isomorphism $\mathcal{T}_X^1 \cong \mathcal{O}_D(X)$ and in particular, if X is d-semistable, then $\mathcal{T}_X^1 \cong \mathcal{O}_D$ [F2, Prop. 2.3].

3.2. Lemma. *There is an exact sequence*

$$0 \longrightarrow \mathcal{T}_X^0 \longrightarrow n_*\Theta_{\widetilde{X}}(\log \bar{D}) \longrightarrow \mathcal{T}_{\bar{D}}^0 \longrightarrow 0\,.$$

Proof. This is a local computation. The sheaf $\Theta_M(\log X)$ of vector fields on M which preserve $z_1z_2z_3 = 0$ is generated by the $z_i\frac{\partial}{\partial z_i}$. Restricted to a component X_i: $z_i = 0$ we get sections of $\Theta_{X_i}(\log D_i)$. The restrictions to different components satisfy the obvious compatibility condition. □

Sections of $\mathcal{T}_D^0$ are given by vector fields on each component, which vanish in the triple points. We study $\Theta_{X_i}(\log D_i)$ with the exact sequence

$$0 \longrightarrow \Theta_{X_i}(\log D_i) \longrightarrow \Theta_{X_i} \longrightarrow \bigoplus_j N_{D_{ij}/X_i} \longrightarrow 0\,.$$

For a d-semistable $K3$-surface X in (-1)-form,

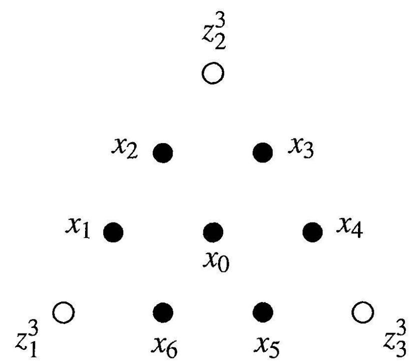
$$H^0(D_{ij}, N_{D_{ij}/X_i}) = H^1(D_{ij}, N_{D_{ij}/X_i}) = 0\,.$$

Each component X_i is $\mathbf{P}^2$ blown up in $k \geq 3$ points and $H^2(\Theta_{X_i}) = 0$, $h^0(\Theta_{X_i}) = \max(0, 8-2k)$, $h^1(\Theta_{X_i}) = \max(0, 2k-8)$.

So $H^0(\Theta_{X_i}) \neq 0$ only in the case that $k=3$ and the double curve D_i is a hexagon. We then call X_i a hexagonal component, or hexagon for short.

3.3. Lemma [F1, Cor. 3.5]. *For a d-semistable $K3$-surface X of type III in (-1)-form, $H^0(X, \mathcal{T}_X^0) = 0$.*

Proof. We first describe the sections of $H^0(\Theta_{X_i})$ for a hexagonal component. We blow up $\mathbf{P}^2$ in the vertices of the coordinate triangle. As basis for the linear system of cubics we take the monomials given by black dots in the picture below.



A vector field ϑ on X_i comes from a vector field on $\mathbf{P}^2$ which vanishes in the points blown up. We can give it homogeneously by $a_1 z_1 \frac{\partial}{\partial z_1} + a_2 z_2 \frac{\partial}{\partial z_2} + a_3 z_3 \frac{\partial}{\partial z_3}$, subject to the relation $z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} = 0$. In the x_j coordinates we get

$$\begin{aligned}(a_1+a_2+a_3)x_0\frac{\partial}{\partial x_0} + (2a_1+a_2)x_1\frac{\partial}{\partial x_1} + (2a_2+a_1)x_2\frac{\partial}{\partial x_2} \\ + (2a_2+a_3)x_3\frac{\partial}{\partial x_3} + (2a_3+a_2)x_4\frac{\partial}{\partial x_4} \\ + (2a_3+a_1)x_5\frac{\partial}{\partial x_5} + (2a_1+a_3)x_6\frac{\partial}{\partial x_6}\,.\end{aligned}$$

We restrict to the line $(x_{j-1}:x_j)$ and take as generator of $\mathcal{T}^0_D|_{D_{ij}}$ the vector field $\vartheta_j = \frac{1}{2}(x_j\frac{\partial}{\partial x_j} - x_{j-1}\frac{\partial}{\partial x_{j-1}})$. On the $(x_6:x_1)$-line $\vartheta = (a_2 - a_3)\vartheta_1$ and on the $(x_1:x_2)$-line $\vartheta = (a_2 - a_1)\vartheta_2$. The remaining coefficients $\vartheta = \beta_j\vartheta_j$ are found by cyclic permutation. They satisfy $\beta_j = \beta_{j-1} + \beta_{j+1}$. In particular, two adjacent coefficients determine all the others and opposite coefficients add up to zero.

Let $\vartheta \in H^0(X, \mathcal{T}^0_X)$ be a non-vanishing global section. As the dual graph is a triangulation of S^2 one has $\sum_i(6 - e_i) = 12$, where e_i is the number of components of the double curve D_i. So there exist non-hexagonal components, and ϑ vanishes on them. Suppose ϑ vanishes on X_0 and not on the adjacent hexagon X_1. We are going to look at the restriction of ϑ to other components, as illustrated in Figure 3.1.

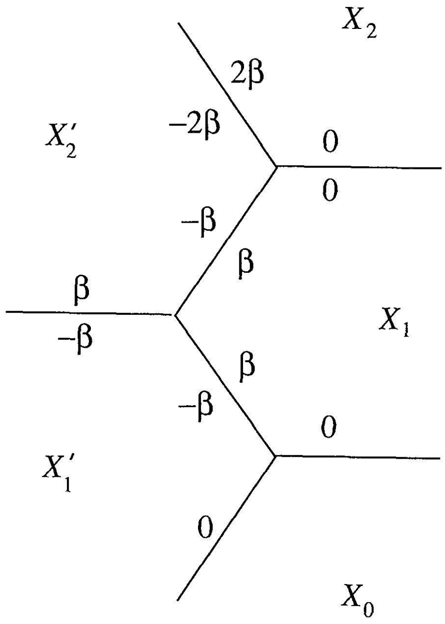


FIGURE 3.1
Coefficients of a vector field

Let $T = X_0 \cap X_1 \cap X_1'$ be a triple point. We know that ϑ vanishes on $X_1 \cap X_0$. If it also vanishes on $X_1 \cap X_1'$, then it vanishes altogether, contrary to the assumption. Therefore X_1' is also hexagonal. Let $\vartheta = \beta\vartheta_0$ on $D_{11'} = X_1 \cap X_1' \subset X_1$. Considered on X_1' the restriction of ϑ is $-\beta$ times the generator. The other triple point on $D_{11'}$ involves a hexagon X_2', which contains also the triple point $X_1 \cap X_2' \cap X_2$. Considered on X_2', the coefficient of the restriction of ϑ to $X_2' \cap X_1'$ is β, to $X_2' \cap X_1$ it is $-\beta$, so to $X_2' \cap X_2$ it is -2β. Therefore on X_2, ϑ has adjacent coefficients 0, 2β. Inductively we find components X_n', X_n with the coefficient $n\beta$ occurring. As there are only finitely many components, this is impossible. □

3.4. THEOREM. *Let* $X = \bigcup_{i=1}^{k} X_i$ *be a* d*-semistable* $K3$*-surface of type III in* (-1)*-form, with* k *components. Then*

$$\begin{aligned} \dim H^1(X, \mathcal{T}_X^0) &= k + 18\,, \\ \dim H^0(X, \mathcal{T}_X^1) &= 1\,, \\ \dim H^1(X, \mathcal{T}_X^1) &= k - 1\,. \end{aligned}$$

So $\dim T_X^1 = k + 19$, $\dim T_X^2 = k - 1$.

Proof. As the dual graph triangulates S^2 we have $V - E + F = 2$, where $V = k$, the number of components of X, E is the number of double curves and F is the number of triple points. Each double curve contains two triple points, so $F = 2/3E$, which makes $E = 3k - 6$. A component X_i, which is $\mathbf{P}^2$ blown up in δ_i points, has $e_i = 9 - \delta_i$ double curves. Observe that $\sum_i e_i = 2E$. The exact sequence above gives $\dim H^1(X, \mathcal{T}_X^0) = \sum_i 2(5 - e_i) + E = 10V - 3E = k + 18$.

We have $h^0(X, \mathcal{T}_X^1) = h^0(D, \mathcal{O}_D) = 1$ and $h^1(X, \mathcal{T}_X^1) = h^1(D, \mathcal{O}_D) = 1 - \chi = 1 - (E - 2F) = k - 1$. □

3.5. Locally trivial deformations of a d-semistable $K3$-surface X are unobstructed and fill up a codimension one smooth subspace of the base of the versal deformation with tangent space $H^1(X, \mathcal{T}_X^0)$. This means that every equation of the base is divisible by the equation of this hypersurface. As one obtains the base space as fibre of a map $T^1 \to T^2$, we look at the map

$$\mathrm{Ob}\colon H^1(\mathcal{T}_X^0) \times H^0(\mathcal{T}_X^1) \to H^1(\mathcal{T}_X^1)\,.$$

Let ξ be a global generator of $\mathcal{T}_X^1$. The existence of a second smooth component (of dimension 20) follows, if one can show that the linear map $\mathrm{Ob}(\ .\ ,\xi)\colon H^1(\mathcal{T}_X^0) \to H^1(\mathcal{T}_X^1)$ is surjective. To describe it we start with the map $\mathrm{Ob}(\ .\ ,\xi)\colon \mathcal{T}_X^0 \to \mathcal{T}_X^1$. Locally X is a hypersurface given by an equation $f = 0$ and elements of $\mathcal{T}_X^0$ come from ambient vector fields satisfying $\vartheta(f) = cf$. We can choose coordinates such that ξ acts as $f \mapsto 1$. Then $\mathrm{Ob}(\vartheta, \xi) = -c\xi$. In the normal crossings situation the map $\mathrm{Ob}(\ .\ ,\xi)$ is surjective and we get an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{T}_X^0 \longrightarrow \mathcal{T}_X^1 \longrightarrow 0\,.$$

The kernel of the map $\mathrm{Ob}(\,.\,,\xi)\colon H^1(\mathcal{T}_X^0) \to H^1(\mathcal{T}_X^1)$ can be characterised in a different way ([F-S]). If $X = \bigcup_{i=1}^{k} X_i$ occurs as central fibre in a degeneration $\mathcal{X} \to S$, we define k line bundles $L_i := \mathcal{O}_{\mathcal{X}}(X_i)|_X$. On a d-semistable X they can be defined by

$$L_i|_{X_i} = \mathcal{O}_{X_i}(-D_i)\,,$$
$$L_i|_{X_j} = \mathcal{O}_{X_j}(X_i \cap X_j)\,, \qquad j \neq i\,,$$

with appropriate gluings, using the global section of $\mathcal{O}_D(X)$. The bundle L_i defines a class ξ_i in

$$H^1(X, \mathcal{O}_X^*) \cong \ker\left\{H^2(X, \mathbf{Z}) \to H^2(\mathcal{O}_X) = \mathbf{C}\right\},$$

which therefore lies in $H^1(\Omega^1/\tau^1)$, where Ω^1/τ^1 are the Kähler differentials modulo torsion [F2, Sect. 1]. The condition that L_i lifts to line bundles on a locally trivial deformation with tangent vector $\vartheta \in H^1(\mathcal{T}_X^0)$ is that $\langle \vartheta, \xi_i \rangle = 0$ with $\langle -,- \rangle$ the perfect pairing $H^1(\mathcal{T}_X^0) \otimes H^1(\Omega^1/\tau^1) \to H^2(\mathcal{O}_X) = \mathbf{C}$ [F2, (2.10)]. The surjectivity of the map $\mathrm{Ob}(\;.\;,\xi)$ follows from the following lemma.

3.6. Lemma. *The classes ξ_i span a $(k-1)$-dimensional subspace of $H^2(X, \mathbf{Z})$.*

Proof. We compute $H^2(X, \mathbf{Z})$ as the kernel of the map

$$\bigoplus H^2(X_i, \mathbf{Z}) \to \bigoplus H^2(D_{ij}, \mathbf{Z})\,.$$

Each ξ_i gives rise to a divisor $\sum_m a_{lm} D_{lm}$ on X_l, $l = 1, \ldots, k$, with coefficients satisfying $a_{lm} + a_{ml} = 0$ (and $a_{lm} \neq 0$ only if $i = l$ or $i = m$). The relation $\sum \xi_i = 0$ holds.

Let now $\sum b_i \xi_i = 0 \in H^2(X, \mathbf{Z})$. It gives rise to a divisor $\sum_m \beta_{lm} D_{lm}$ on X_l. If the classes D_{lm} are independent in $H^2(X_l, \mathbf{Z})$, then $\beta_{lm} = 0$ for all m. This condition is not satisfied if X_l is a hexagon. Then we can only conclude that $\beta_{l,m-1} + \beta_{l,m+1} = \beta_{lm}$. With the same argument as in the proof of Theorem 3.3, illustrated by Figure 3.1, we infer that even in this case $\beta_{lm} = 0$ for all m.

Therefore $b_i = b_j$ for all pairs (i,j) such that $X_i \cap X_j \neq \varnothing$. This implies that $\sum b_i \xi_i$ is a multiple of $\sum \xi_i$. □

We summarise:

3.7. THEOREM [F2, (5.10)]. *A d-semistable $K3$-surface X of type III is smoothable. Its versal base space is the union $V_1 \cup V_2$, where V_1 is a smooth hypersurface corresponding to locally trivial deformations of X, which meets transversally a 20-dimensional smooth subspace V_2, with $V_2 \setminus V_1$ parametrising smooth $K3$-surfaces and $V_2 \cap V_1$ locally trivial deformations of X for which $\mathcal{O}_D(X)$ remains trivial.*

3.8. EMBEDDED DEFORMATIONS. We relate the above results to direct computations with generators and relations for the cone over X, as for the tetrahedron. The case of cones over non-singular varieties is treated in [S2]. We suppose that the affine cone $C(X)$ over X is Cohen-Macaulay. The starting point is the exact sequence

$$0 \longrightarrow T^0_{C(X)} \longrightarrow \Theta_{\mathbf{C}^{n+1}}|_{C(X)} \longrightarrow N_{C(X)} \longrightarrow T^1_{C(X)} \longrightarrow 0, \tag{3.1}$$

which we shall relate to exact sequences of sheaves on X. We set $U = C(X)\setminus 0$; then $\pi\colon U \to X$ is a $\mathbf{C}^*$-bundle over X. For a reflexive sheaf $\mathcal{F}$ on $C(X)$ we have $H^0(C(X), \mathcal{F}) = H^0(U, \mathcal{F})$. All sheafs $\mathcal{F}$ considered here have a natural $\mathbf{C}^*$-action, so $\pi_*\mathcal{F}$ decomposes into the direct sum of eigenspaces. In particular, the degree 0 part is the sheaf of $\mathbf{C}^*$-invariants. With homogeneous coordinates x_i the $\mathbf{C}^*$-invariant sections $x_j\frac{\partial}{\partial x_i}$ of $H^0(U, \Theta_{\mathbf{C}^{n+1}}|_{C(X)})$ can be considered as elements of $H^0(X, V^* \otimes_{\mathbf{C}} \mathcal{O}_X(1))$, where $V = H^0(X, \mathcal{O}_X(1))$. We get the degree zero part $T^1_{C(X)}(0)$ as $\operatorname{coker} H^0(X, V^* \otimes_{\mathbf{C}} \mathcal{O}_X(1)) \to H^0(X, N_{X/\mathbf{P}^n})$. We factorise this map corresponding to a splitting of the exact sequence (3.1):

$$0 \longrightarrow T^0_{C(X)} \longrightarrow \Theta_{\mathbf{C}^{n+1}}|_{C(X)} \longrightarrow G \longrightarrow 0, \tag{3.2}$$

$$0 \longrightarrow G \longrightarrow N_{C(X)} \longrightarrow T^1_{C(X)} \longrightarrow 0. \tag{3.3}$$

Denoting by $\mathcal{G}_X$ the sheaf of $\mathbf{C}^*$ invariants associated to G we obtain

$$H^0(X, V^* \otimes_{\mathbf{C}} \mathcal{O}_X(1)) \longrightarrow H^0(X, \mathcal{G}_X) \longrightarrow H^0(X, N_{X/\mathbf{P}^n}).$$

On X we have the exact sequence

$$0 \longrightarrow \mathcal{G}_X \longrightarrow N_{X/\mathbf{P}^n} \longrightarrow \mathcal{T}^1_X \longrightarrow 0.$$

The short exact sequence (3.2) gives

$$0 \longrightarrow \mathcal{D}iff_X \longrightarrow V^* \otimes_{\mathbf{C}} \mathcal{O}_X(1) \longrightarrow \mathcal{G}_X \longrightarrow 0$$

with $\mathcal{D}iff_X$ the sheaf of differential operators on X, which is related to $\mathcal{T}^0_X$ by the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}iff_X \longrightarrow \mathcal{T}^0_X \longrightarrow 0.$$

3.9. PROPOSITION. *Let X be a d-semistable $K3$ of type III in (-1)-form. The space of infinitesimal locally trivial embedded deformations is $H^1(X, \mathcal{D}iff_X)$, of dimension $k+17$. It has codimension one in $T^1_{C(X)}(0)$.*

Proof. From the computation of $h^i(\mathcal{T}^0_X)$ in 3.4 and the exact sequence for $\mathcal{D}iff_X$ we conclude that $h^0(\mathcal{D}iff_X) = h^0(\mathcal{O}_X) = 1$. As $h^1(\mathcal{O}_X) = 0$ and $h^2(\mathcal{O}_X) = 1$ we get the exact sequence

$$0 \longrightarrow H^1(\mathcal{D}iff_X) \longrightarrow H^1(\mathcal{T}^0_X) \longrightarrow H^2(\mathcal{O}_X) \longrightarrow H^2(\mathcal{D}iff_X) \longrightarrow 0.$$

The line bundle $\mathcal{O}(1)$ determines a class $h \in H^1(\Omega^1/\tau^1)$, which lifts to a deformation $\vartheta \in H^1(X, \mathcal{T}^0_X)$ if and only if $\langle \vartheta, h \rangle = 0$ with $\langle -, - \rangle$ the perfect pairing $H^1(\mathcal{T}^0_X) \otimes H^1(\Omega^1/\tau^1) \to H^2(\mathcal{O}_X) = \mathbf{C}$. This accounts for the non-algebraic deformation direction. So $\dim H^1(\mathcal{D}iff_X) = k+17$ and $H^2(\mathcal{D}iff_X) = 0$. We then obtain

$$H^1(X, \mathcal{D}iff_X) = \operatorname{coker}\left\{H^0(X, V^* \otimes_{\mathbf{C}} \mathcal{O}_X(1)) \longrightarrow H^0(X, \mathcal{G}_X)\right\}$$

and $h^1(\mathcal{G}_X) = 0$, as $h^i(X, \mathcal{O}_X(1)) = 0$ for $i > 0$. Finally we get $H^1(N_{X/\mathbf{P}^n}) = H^1(\mathcal{T}^1_X)$ and the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{G}_X) \longrightarrow H^0(X, N_{X/\mathbf{P}^n}) \longrightarrow H^0(X, \mathcal{T}^1_X) \longrightarrow 0. \qquad \square$$

3.10. For $T^2_{C(X)}(0)$ we can argue as in the smooth case [S2, (1.25)] to obtain the exact sequence

$$0 \longrightarrow T^2_{C(X)}(0) \longrightarrow H^1(X, N_{X/\mathbf{P}^n}) \longrightarrow \bigoplus H^1(X, \mathcal{O}_X(d_j))$$

with the d_j the degrees of the generators of the ideal of $C(X)$ (or of X). In particular, in our situation $T^2_{C(X)}(0) = H^1(N_{X/\mathbf{P}^n}) = H^1(\mathcal{T}^1_X)$.

3.11. THEOREM [F-S, (5.5)]. *A d-semistable $K3$-surface X of type III in $\mathbf{P}^n$ is smoothable by embedded deformations. They form a 19-dimensional smooth component.*

Proof. In the embedded case the base space is also the fibre of a map between the relevant cotangent modules, and the locally trivial deformations are unobstructed. The map $\mathrm{Ob}\colon H^1(\mathcal{D}iff_X) \times H^0(\mathcal{T}^1_X) \to H^1(\mathcal{T}^1_X)$ is the restriction of the obstruction map in 3.5. We observe that $H^1(\mathcal{D}iff_X)$ is transversal to $\bigcap_i \ker \mathrm{Ob}(\,.\,, \xi_i)$, as the class h satisfies $h^2 > 0$ and is therefore independent of the classes of the ξ_i. $\square$

3.12. THE TOPOLOGY OF THE SPECIAL FIBRE. One can compute the homology $H_*(X,\mathbf{Z})$ with a Mayer-Vietoris spectral sequence [P, Prop. 2.5.1] with E^1-term $E^1_{p,q} = H_p(X^{[q]},\mathbf{Z})$, where $X^{[0]} = \coprod X_i$, $X^{[1]} = \coprod D_{ij}$ and $X^{[2]}$ the set of triple points $P_{ijk} = X_i \cap X_j \cap X_k$.

3.13. PROPOSITION. *Let* $X = \bigcup_{i=1}^{k} X_i$ *be a* d*-semistable* $K3$*-surface of type III in* (-1)*-form, with* k *components. Then*

$$\dim H_0(X,\mathbf{Z}) = 1\,,$$
$$\dim H_2(X,\mathbf{Z}) = k+19\,,$$
$$\dim H_4(X,\mathbf{Z}) = k\,.$$

Proof. The E^1-term of the spectral sequence looks like:

$$\begin{array}{|lll}
\bigoplus H_4(X_i,\mathbf{Z}) & & \\
\quad 0 & & \\
\bigoplus H_2(X_i,\mathbf{Z}) & \bigoplus H_2(D_{ij},\mathbf{Z}) & \\
\quad 0 & \quad 0 & \\
\bigoplus H_0(X_i,\mathbf{Z}) & \bigoplus H_0(D_{ij},\mathbf{Z}) & \bigoplus H_0(T_{ijk},\mathbf{Z}) \\
\hline
\end{array}$$

To prove that the map $\bigoplus H_2(D_{ij},\mathbf{Z}) \to \bigoplus H_2(X_i,\mathbf{Z})$ is injective we observe that $\bigoplus_j H_2(D_{ij},\mathbf{Z}) \to H_2(X_i,\mathbf{Z})$ is injective unless X_i is a hexagonal component. We take care of those by arguing as in the proofs of Lemmas 3.3 and 3.6. If the component X_i is obtained by blowing up $\mathbf{P}^2$ in δ_i points, then $b_2(X_i) = \delta_i + 1 = 10 - e_i$ with the notation of 3.3, so the cokernel of the map $\bigoplus H_2(D_{ij},\mathbf{Z}) \to \bigoplus H_2(X_i,\mathbf{Z})$ has dimension $10V - 3E = k + 18$. The dimension formulas now follow from the spectral sequence. □

3.14. We describe the non-algebraic homology class in more detail. Each double curve contains two triple points, which are homologous, so the boundary of an interval. On a component X_i these intervals make up a closed polygon (with e_i edges), which itself is the boundary of a topological disc. For the case of $\mathbf{P}^2$ blown up in 4 points this is illustrated in Figure 5.1: after blowing up we have a pentagon, which is the boundary of the strict transform of the shaded area. With the given coordinates this strict transform consists of all points on the Del Pezzo surface with positive coordinates. Finally the discs glue together to a real polyhedron with the same dual graph as the complex surface X.

3.15. A nice construction for studying the homology of the general fibre is given by [A'C]. Let $\sigma_i\colon \mathcal{Z}_i \to \mathcal{X}$ be the oriented real blow-up of $X_i \subset \mathcal{X}$. This is a manifold with boundary, whose boundary $\partial \mathcal{Z}_i = \sigma_i^{-1}(X_i)$ is isomorphic to the boundary of a tubular neighbourhood of X_i in $\mathcal{X}$. The fibred product $\sigma\colon \mathcal{Z} \to \mathcal{X}$ of the σ_i is a manifold with corners. Its boundary $\mathcal{N} := \partial \mathcal{Z}$ comes with a map to X. It also fibres over S^1 : the composed map $\mathcal{Z} \to \mathcal{X} \to S \ni 0$ extends to a map from $\mathcal{Z}$ to the real oriented blow-up of S in 0 (polar coordinates !). A fibre of $\mathcal{N} \to S^1$ is then a topological model of the general fibre.

This model is not sufficient to describe the monodromy. One has first to replace X by the geometric realisation of the simplicial object $X^{[\cdot]}$: one replaces each double point by an interval, and each triple point by a 2-simplex. A final fibred product then gives the new model. For details see [A'C, §2].

4. Hodge algebras

4.1. Stanley–Reisner rings. Let Δ be a simplicial complex with set of vertices $V = \{v_1, \dots, v_n\}$. A monomial on V is an element of $\mathbf{N}^V$. Each subset of V determines a monomial on V by its characteristic function. The support of a monomial $M\colon V \to \mathbf{N}$ is the set $\operatorname{supp} M = \{v \in V \mid M(v) \neq 0\}$. The set Σ_Δ of monomials whose support is not a face is an ideal, generated by the monomials corresponding to minimal non-simplices.

Given a ring R and an injection $\phi\colon V \to R$ we can associate to each monomial M on V the element $\phi(M) = \prod_{v\in V} \phi(v)^{M(v)} \in R$. We will usually identify V and $\phi(V)$ and write $M \in R$ for $\phi(M)$. This applies in particular to the polynomial ring $K[V]$ over a field K. The ideal Σ_Δ gives rise to the Stanley-Reisner ideal $I_\Delta \subset K[V]$. The *Stanley-Reisner ring* is $A_\Delta = K[V]/I_\Delta$.

Deformations of Stanley-Reisner rings are studied in [A-C].

4.2. Example. Let Δ be an octahedron. We map the set of vertices to $\mathbf{C}[x_1, \dots, x_6]$ such that opposite vertices correspond to variables with index sum 7.

The Stanley-Reisner ring is minimally generated by the three monomials $x_i x_{7-i}$. The spaces smoothes to a $K3$-surface, the complete intersection of three general quadrics. A general 1-parameter deformation is not semi-stable, because the total space has singularities at the six quadruple points of the special fibre.