5. The dodecahedron

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4.5. If R is graded and the elements of $\phi(H)$ are homogeneous the straightening relations give a presentation for R [D-E-P, p. 15].

We note that R is a deformation of the discrete Hodge algebra governed by Σ, whose ideal is generated by the monomials M.

4.6. EXAMPLE. The equations of the tetrahedron of degree 12 of 2.2 are straightening relations. We take Σ as Stanley-Reisner ideal Σ_Δ, where Δ is the stellation of the tetrahedron: in each top-dimensional face we take an additional vertex, which is joined to all vertices on the face. The partial order on the set of vertices is obtained by declaring the new vertices to be smaller. The discrete Hodge algebra then has equations x_ix_j, x_iy_j and $y_jy_ky_l$.

5. THE DODECAHEDRON

5.1. To get an icosahedron as dual graph we need the incidence relations of a dodecahedron. Each side should be a rational surface and the intersection with the other surfaces should have a pentagon as dual graph. A pentagon occurs as hyperplane section of a Del Pezzo surface of degree 5. So we can realise our dodecahedron by gluing together 12 Del Pezzo surfaces.

We first describe the Del Pezzo surfaces. Each of those is an extension of its pentagonal hyperplane section. Its coordinate ring can be obtained as Stanley-Reisner ring of a pentagon as 1-dimensional simplicial complex. Introducing variables y_i, we get the equations $y_{i-1}y_{i+1}$. With an extra variable x the Del Pezzo surface has equations

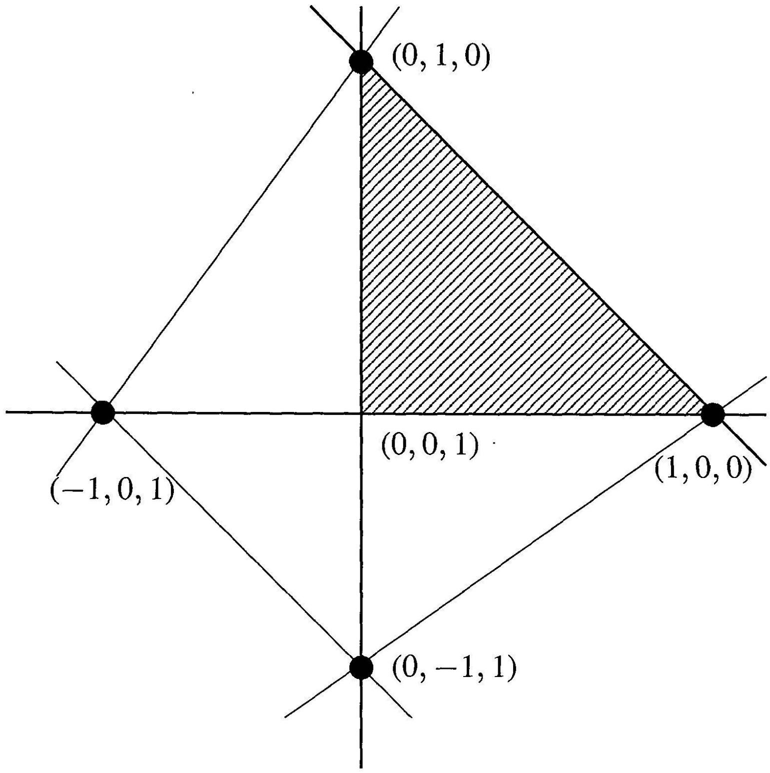
$$y_{i-1}y_{i+1} - xy_i - x^2 .$$

These are the Pfaffians of the matrix

$$\begin{pmatrix} 0 & y_1 & x & -x & -y_5 \\ -y_1 & 0 & y_2 & x & -x \\ -x & -y_2 & 0 & y_3 & x \\ x & -x & -y_3 & 0 & y_4 \\ y_5 & x & -x & -y_4 & 0 \end{pmatrix} .$$

We can check that this is indeed a smooth Del Pezzo of degree 5 by giving an explicit birational map from $\mathbf{P}^2$, which blows up four points, see Figure 5.1.

To the variable x corresponds a new vertex at the centre of the pentagon. By joining it to all other vertices we obtain a 2-dimensional simplicial complex, and the homogeneous coordinate ring of the Del Pezzo surface is a graded Hodge algebra governed by the Stanley-Reisner ideal of the complex: to satisfy H–2 we take x to be less than all y_i.



$$
\begin{aligned}
y_1 &= v^2 u \\
y_2 &= vw(v+w) \\
y_3 &= w^2(u+v+w) \\
y_4 &= uw(u+w) \\
y_5 &= u^2 v \\
x &= uvw
\end{aligned}
$$

FIGURE 5.1
Blowing up $\mathbf{P}^2$ in 4 points

5.2. To construct a normal crossings dodecahedron of degree 60 we glue twelve Del Pezzo surfaces. We get a simplicial complex Δ by stellating a dodecahedron: we take in each face the centre of the pentagon as extra vertex. A non-convex realisation of this complex is the great stellated dodecahedron.

5.3. PROPOSITION. *The coordinate ring of the dodecahedron of degree* 60 *is a graded Hodge algebra governed by* Σ_Δ.

We describe the equations for a dodecahedron X with icosahedral symmetry in more detail. We have 20 variables y_α, one for each dodecahedral vertex and 12 variables x_i from the extra vertices in the faces. We will denote the vertices by α and i. As two face vertices are not connected by an edge we have 66 equations $x_i x_j = 0$, $i \neq j$. If $\overline{i\alpha}$ is not an edge, we have $x_i y_\alpha = 0$; there are 180 such equations. The non-edges $\overline{\alpha\beta}$ come in two types: in 100 cases the line $\overline{\alpha\beta}$ does not lie in a face of the original dodecahedron, leading to $y_\alpha y_\beta = 0$; if it lies in such a face we get a Del Pezzo equation (in $5 \times 12 = 60$ cases).

We summarise:

Type	Equation	Conditions	#
(1)	$x_i x_j$	$i \neq j$	66
(2)	$x_i y_\alpha$	$\overline{i\alpha}$ not an edge	180
(3)	$y_\alpha y_\beta$	$\overline{\alpha\beta}$ not in face	100
(4)	$y_\alpha y_\gamma - x_i y_\beta - x_i^2$	Del Pezzo	60

The relations follow from the relations between the generators of the Stanley-Reisner ideal of the great stellated dodecahedron, which have a particularly simple form: we get a relation for each pair of equations which have one variable in common.

This gives the following list of relations, where we suppress the conditions on the indices; they can be deduced from the list of equations.

Type	Relation	#
(1–1)	$(x_i x_j)\, x_k - (x_i x_k)\, x_j$	440
(2–1)	$(x_i y_\alpha)\, x_j - (x_i x_j) y_\alpha$	1980
(2–2)	$(x_i y_\alpha) y_\beta - (x_i y_\beta) y_\alpha$	1260
(3–2)	$(y_\alpha y_\beta)\, x_i - (x_i y_\alpha) y_\beta$	1200
(3–3)	$(y_\alpha y_\beta) y_\gamma - (y_\alpha y_\gamma) y_\beta$	780
(4–2)	$(y_\alpha y_\gamma - x_i y_\beta - x_i^2)\, x_j - (x_j y_\alpha) y_\gamma + (x_i x_j)(x_i + y_\beta)$	660
(4–3)	$(y_\alpha y_\gamma - x_i y_\beta - x_i^2) y_\delta - (y_\alpha y_\delta) y_\gamma + (x_i y_\delta)(x_i + y_\beta)$	860
(4–3)	$(y_\alpha y_\gamma - x_i y_\beta - x_i^2) y_\delta - (y_\alpha y_\delta - x_j y_\beta - x_j^2) y_\gamma$	
	$+ (x_i y_\delta)(x_i + y_\beta) - (x_j y_\gamma)(x_j + y_\beta)$	40
(4–4)	relations from matrices	60

We use the equations and relations to compute infinitesimal deformations. The computations are similar to the case of the tetrahedron of degree 12. To illustrate our methods we prove that the dodecahedron X has no nontrivial extensions. This statement means that X is only a hyperplane section of the projective cone over it. To prove this we have to show that the affine cone $C(X)$ has no deformations of negative degree.

5.4. PROPOSITION. *$T^1_{C(X)}(-\nu) = 0$ for $\nu > 0$.*

Proof. As all quadratic equations occur in linear relations we cannot perturb the equations with constants, so $T^1(-2) = 0$.

Now we consider deformations of degree -1. We start with equations of type (1), which we perturb as follows:

$$x_i x_j + \sum a_{ij}^m x_m + \sum b_{ij}^\alpha y_\alpha\,.$$

The relations (1–1) together with the equations give

$$a_{ij}^k x_k^2 + \sum_{\overline{k\alpha}\text{ edge}} b_{ij}^\alpha x_k y_\alpha = a_{ik}^j x_j^2 + \sum_{\overline{j\alpha}\text{ edge}} b_{ik}^\alpha x_j y_\alpha .$$

This shows that $a_{ij}^k = 0$ for $k \notin \{i,j\}$. For each α we can find a $k \notin \{i,j\}$ such that $\overline{k\alpha}$ is an edge, so $b_{ij}^\alpha = 0$ and the deformation has the form

$$x_i x_j + a_{ij}^i x_i + a_{ij}^j x_j .$$

We now perturb equations of type (2):

$$x_i y_\alpha + \sum a_{i\alpha}^m x_m + \sum b_{i\alpha}^\beta y_\beta$$

and use the relations of type (2–1):

$$a_{i\alpha}^j x_j^2 + \sum_{\overline{j\beta}\text{ edge}} b_{i\alpha}^\beta x_j y_\beta = a_{ij}^j x_j y_\alpha$$

to conclude that $a_{i\alpha}^j = 0$ for all $j \neq i$, $b_{i\alpha}^\beta = 0$ for all $\beta \neq \alpha$ and $b_{i\alpha}^\alpha = a_{ij}^j$ for all j such that $\overline{j\alpha}$ is an edge. It follows that $a_{ij}^j = a_{ik}^k$ for all j and k. Using the coordinate transformation ∂_{x_i} we may therefore assume that the equations of type (1) are not perturbed at all, while those of type (2) have the form $x_i y_\alpha + a_{i\alpha}^i x_i$.

Perturbing equations of type (3) in a similar manner as $y_\alpha y_\beta + \sum a_{\alpha\beta}^m x_m + \sum b_{\alpha\beta}^\gamma y_\gamma$ we find from the relations (3-2) that

$$a_{\alpha\beta}^i x_i^2 + \sum_{\overline{i\gamma}\text{ edge}} b_{\alpha\beta}^\gamma x_i y_\gamma = a_{i\alpha}^i x_i y_\beta .$$

So $a_{\alpha\beta}^i = 0$, $b_{\alpha\beta}^\gamma = 0$ for $\gamma \notin \{\alpha,\beta\}$ and $b_{\alpha\beta}^\beta = a_{i\alpha}^i$. The coordinate transformation ∂_{y_α} can be used to eliminate the perturbation of the equations of type (2). Then those of type (3) are not perturbed either.

Finally we look at the Del Pezzo equations (4). From the relations (4–2) we conclude as before that the only possible perturbations have the form

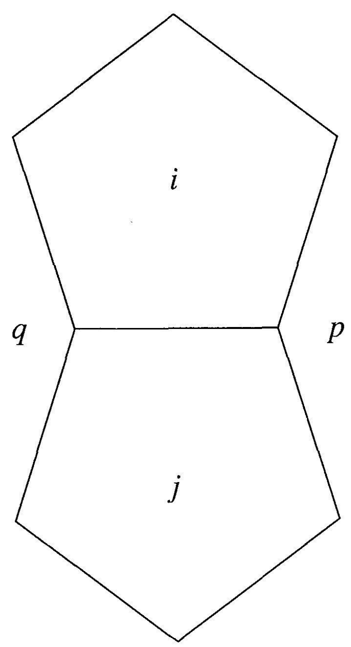
$$y_\alpha y_\gamma - x_i y_\beta - x_i^2 + a_{\alpha\gamma}^i x_i .$$

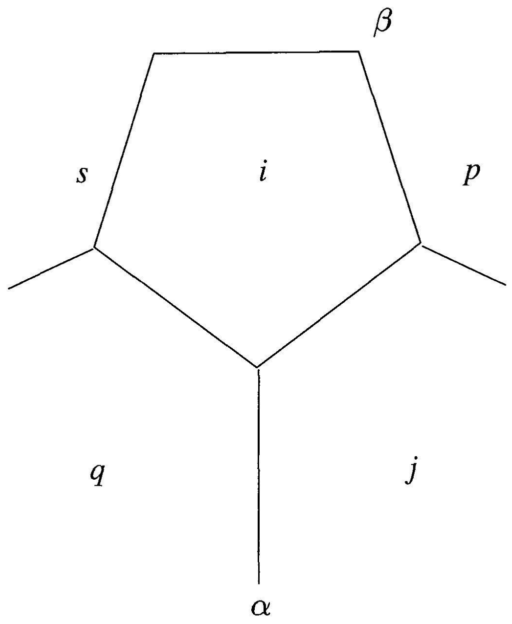
Here $\overline{i\alpha}$, $\overline{i\beta}$ and $\overline{i\gamma}$ all are edges. This means that we can look at each Del Pezzo separately. From the matrix of relations we obtain that $a_{\alpha\gamma}^i = 0$. □

5.5. PROPOSITION. *The dodecahedron X with icosahedral symmetry is d-semistable. The space of locally trivial embedded deformations has dimension* 29.

Proof. We describe a global section of the sheaf $\mathcal{T}^1_X$, without proof. To formulate the result we use the alternative notation y_{ijk} for y_α, if $\overline{i\alpha}$, $\overline{j\alpha}$ and $\overline{k\alpha}$ are the edges involving α.

The equations of type (1) are not deformed, unless $\overline{ij}$ is an edge, in which case we have $x_ix_j + dy_{ijp}y_{ijq}$. An equation of type (2) is perturbed to $x_iy_\alpha + d(x_jy_{ijp} + x_jy_{ijq} + x_j^2)$ if $\alpha = (jkl)$ is opposite the edge $[pq]$; if ijp, ijq and $\alpha = jqr$ are three consecutive vertices, as are $\alpha = jqr$, ijq and iqs, we get $x_iy_\alpha + d(x_jy_{ijq} + x_qy_{ijq} + y_{ijq}^2)$ and in all other cases the equation is not deformed.





All perturbations of the equations (3) vanish except when α and β are nearest possible: one can reach β from α by passing three edges. Suppose that the vertices on this path are $\beta = ipo$, ijp, ijq, $jqr = \alpha$. Then we set $y_\alpha y_\beta + d(x_iy_{ijq} + x_jy_{ijp} + y_{ijp}y_{ijq})$. The Del Pezzo equations are not deformed.

We compute locally near a triple point and look at the chart $y_\alpha = 1$. All variables can be eliminated except y_β, y_γ and y_δ such that $\overline{\alpha\beta}$, $\overline{\alpha\gamma}$ and $\overline{\alpha\delta}$ are edges, and x_i, x_j and x_k, where α, β and γ lie on the face k, etc. We have nine equations left, of three types: $x_ix_j + dy_\delta$, $x_ky_\delta + d(1 + x_i + x_j)$ and the Del Pezzo equation $y_\beta y_\gamma - x_k - x_k^2$. The last one shows that even x_k can be eliminated, as the double curve lies in $x_k = 0$. By multiplying the Del Pezzo equation with y_δ and using the other equations we get

$$y_\beta y_\gamma y_\delta + d(1 + x_i + x_j + x_k)\,.$$

This shows that our d-deformation indeed represents the class $[1] \in H^0(\mathcal{O}_D) = H^0(\mathcal{T}^1_X)$. $\square$

Proposition 3.9 gives the dimension of the space of locally trivial deformations, but it can also be computed directly. For each Del Pezzo we have 5 deformations by multiplying the x_i in a given column of the defining matrix with a unit of the form $1+\varepsilon_{ij}$. These deformations are trivial and can also be obtained by multiplying the y_α by suitable factors. In total we have 60 such deformations, but globally we have only 31 diagonal coordinate transformations (32 variables, but we have to subtract one for the Euler vector field).

5.6. THEOREM. *There exists a semistable degeneration of $K3$-surfaces of degree 60 with icosahedral symmetry, whose special fibre is our dodecahedron.*

5.7. The rotation group $G_{60} \cong A_5$ of the icosahedron acts symplectically on the general fibre $\mathcal{X}_t$ and the quotient $\mathcal{X}_t/G_{60}$ is again a $K3$-surface, with 2 A_4, 3 A_2 and 4 A_1 singularities [X]. The locus of such surfaces has dimension two in moduli, so together with a polarisation there is only a curve of such surfaces. It would be interesting to know this curve. A deformation computation as above only gives a parametrisation with power series; anyway, the computation is too complicated.

We can take the quotient of the special fibre, which is our dodecahedron. Invariants for the icosahedral reflection group are

$$X = \sum x_i\,, \qquad Y_1 = \sum y_\alpha\,, \qquad Y_2 = \sum_{\overline{\alpha\beta}\ \text{edge}} y_\alpha y_\beta\,,$$

and a skew invariant Z is obtained by taking the G_{60} orbit of $x_i y_\alpha y_\beta (y_\alpha - y_\beta)$. After a coordinate transformation $X \mapsto \frac{1}{5}X$, $Y_2 \mapsto Y_2 + \frac{1}{5}XY_1 + \frac{1}{5}X^2$ the quotient is given by the equation

$$\begin{aligned} 0 = Z^2 &+ 5X^2Y_2^2(4Y_2 + 8X^2 + 12XY_1 - Y_1^2) \\ &+ (30Y_1 + 20X)Y_2X^3(X^2 + XY_1 - Y_1^2) + X^3(3X + 4Y_1)(X^2 + XY_1 - Y_1^2)^2 . \end{aligned}$$

This is a surface of degree 8 in the weighted projective space $\mathbf{P}(1,1,2,4)$. These numbers are in Reid's list of famous 95 and the general $X_8 \subset \mathbf{P}(1,1,2,4)$ is a $K3$-surface with 2 A_1 singularities. Our surface has a double line and two A_4 singularities, at $Y_2 = X^2 + XY_1 - Y_1^2 = 0$.

5.8. Finite groups acting symplectically on $K3$-surfaces have been classified by Mukai [Mu], see also [X]. Mukai gives an example of a $K3$-surface with even an action of the symmetric group S_5:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0\,,$$
$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0\,,$$
$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\,.$$

An element of S_5 acts by permuting the coordinates $x_1, \ldots, x_5$ and multiplying x_0 by its sign. By changing the last equations to $tx_0^3+x_1^3+x_2^3+x_3^3+x_4^3+x_5^3 = 0$ we obtain a pencil of A_5-invariant surfaces. The semistable model of the degeneration at $t = \infty$ is of type I.

5.9. THE STANLEY-REISNER RING OF THE ICOSAHEDRON. A different family of $K3$-surfaces with icosahedral symmetry is obtained by smoothing the Stanley-Reisner ring of the icosahedron. The infinitesimal deformations can be found from [A-C, Sect. 4] or computed directly with the methods above. All deformations are unobstructed ($T^2_{C(X)} = 0$). We have $T^1_{C(X)}(\nu) = 0$ for $\nu < 0$ and $\dim T^1_{C(X)}(0) = 30$. Furthermore the dimension of $H^0(\Theta_X)$ equals 11, which fits with the fact that X deforms to smooth $K3$-surfaces ($30 - 11 = 19$).

We number the vertices as in Figure 5.2.

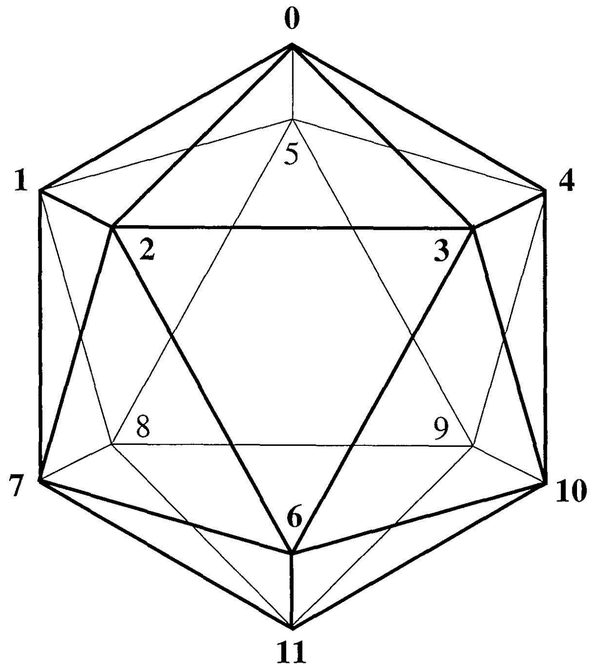


FIGURE 5.2
Icosahedron

We have two types of equations, depending on the distance between vertices. On suitable representatives the infinitesimal deformations are

$$x_0x_6 + \varepsilon_{06}\, x_2x_3\,, \qquad x_0x_{11}\,.$$

By taking all ε_{ij} equal we get an icosahedral invariant deformation. The lift to a one-parameter deformation seems to involve power series of the deformation variable (I computed up to order 7). Anyway, equations for a $K3$ of degree 20 are not very illuminating.

As before, this deformation is not semistable, because the total space has singularities. Each vertex of the icosahedron gives a singularity, which is the cone over a pentagon. It is smoothed negatively, with total space the cone over a Del Pezzo of degree 5. We resolve these singularities by blowing up. We introduce 12 Del Pezzo surfaces. The sides are blown up in three points, giving hexagons. The dual graph of the central fibre is now a stellated dodecahedron. The object itself consists of pentagons and hexagons. It contains a real homology class, as described in 3.14, which looks like a football, so our special fibre is a complexified football.

5.10. A DEGENERATION OF DEGREE 12. The existence of the two type III degenerations above with icosahedral symmetry follows from a deformation argument, but it is too complicated to give explicit equations. In the football case pentagons arise because of the singularities of the total space. This suggests that one can get a degeneration of low degree by blowing down components of the special fibre. Blowing down means removing vertices from the dual graph.

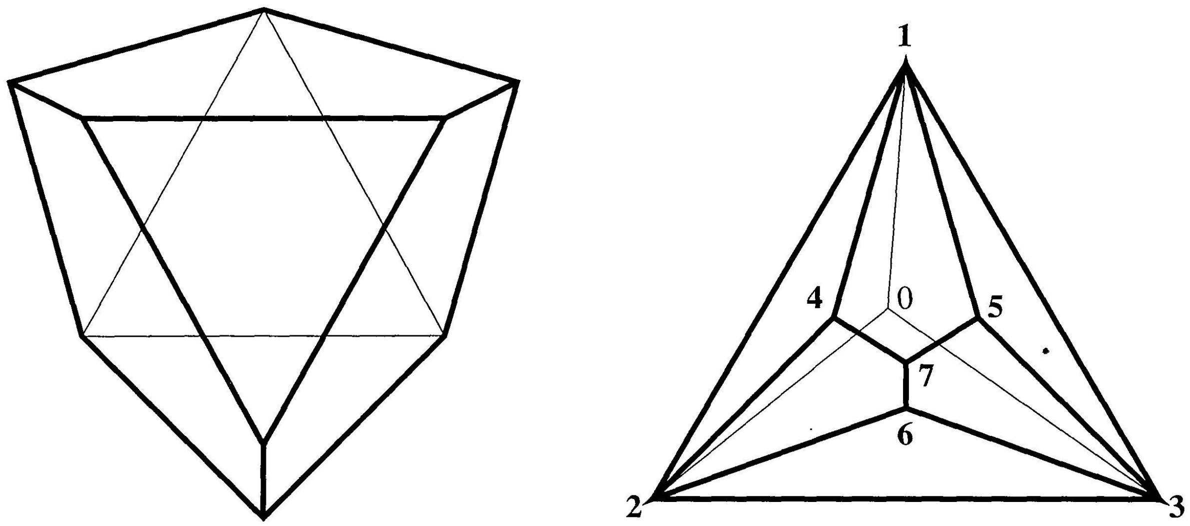


FIGURE 5.3
Dual graph and its realisation for X of degree 12

We start from the icosahedron (Figure 5.2) and remove non-adjacent vertices, say those numbered 0, 7 and 10. This means breaking the symmetry. The resulting dual graph is shown in Figure 5.3. Of the double curves on the components, six are triangles and four are rectangles. We realise them on planes, resp. quadric surfaces. The picture also shows a realisation (as a real polyhedron). We cannot take the Stanley-Reisner ideal, as the realisation contains rectangles. For those we take an equation of the form $xy - zt$. One may think that a rectangle can be triangulated in two ways, each giving a monomial, which are forced to be equal. The result is a surface $X \subset \mathbf{P}^7$ of degree 12. With the numbering in the figure we get the S_3-invariant ideal

$$\begin{gathered} x_0x_7 , \\ x_0x_4 , \quad x_0x_5 , \quad x_0x_6 , \\ x_1x_6 , \quad x_2x_6 , \quad x_3x_4 , \\ x_1x_7 - x_4x_5 , \quad x_2x_7 - x_4x_6 , \quad x_3x_7 - x_5x_6 , \\ x_1x_2x_3 . \end{gathered}$$

The next thing to do is to compute the T^1 and T^2 for the affine cone $C(X)$ over X. This is conveniently done with a computer algebra program. A computation with *Macaulay* [B-S] gives the following result:

5.11. LEMMA. *As $\mathcal{O}_{C(X)}$-module $T^1_{C(X)}$ is generated by eight elements, represented by the following perturbations of the equations:*

$$\begin{gathered} x_0x_7 , \\ x_0x_4 - c_3x_1x_2 , \quad x_0x_5 - c_2x_1x_3 , \quad x_0x_6 - c_1x_2x_3 , \\ x_1x_6 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 , \\ x_2x_5 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 , \\ x_3x_4 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 , \\ x_1x_7 - x_4x_5 , \quad x_2x_7 - x_4x_6 , \quad x_3x_7 - x_5x_6 , \\ x_1x_2x_3 + ax_0 + b_1x_2x_3 + b_2x_1x_3 + b_3x_1x_2 , \end{gathered}$$

and $\dim T^2_{C(X)} = 2$, *concentrated in degree* -2.

The quadratic obstruction is given by $a(c_1 - c_2) = a(c_1 - c_3) = 0$. We conclude that the degree zero deformations are unobstructed. The base space for $C(X)$ in non-positive degrees has two components. As we are mainly interested in S_3-invariant deformations we consider only the component with

$c_1 = c_2 = c_3$ $(=: c)$. The component will be obtained by substituting polynomials for the deformation variables a, b_i and c. A computation gives the equations

$$
\begin{gathered}
x_0x_7 + c(b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac)\,,\\
x_0x_4 - cx_1x_2\,, \quad x_0x_5 - cx_1x_3\,, \quad x_0x_6 - cx_2x_3\,,\\
x_1x_6 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac\,,\\
x_2x_5 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac\,,\\
x_3x_4 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac\,,\\
x_1x_7 - x_4x_5\,, \quad x_2x_7 - x_4x_6\,, \quad x_3x_7 - x_5x_6\,,\\
x_1x_2x_3 + ax_0 + b_1x_2x_3 + b_2x_1x_3 + b_3x_1x_2 - b_0(b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac).
\end{gathered}
$$

For $c \neq 0$ we derive the three equations $x_0x_7 - cx_ix_{7-i}$, which show that we have a hypersurface in the cone over $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

5.12. PROPOSITION. *A general one-parameter deformation in degree 0 on the component described above has a minimal model in (-1)-form with the icosahedron as dual graph for the central fibre. In particular, this holds for*

$$
\begin{aligned}
a &= c(x_0^2 + x_0(x_1 + x_2 + x_3) + x_1^2 + x_2^2 + x_3^2)\,,\\
b_0 &= cx_7\,,\\
b_1 &= c(x_2 + x_3 + x_6 + x_7)\,,\\
b_2 &= c(x_1 + x_3 + x_5 + x_7)\,,\\
b_3 &= c(x_1 + x_2 + x_4 + x_7)\,.
\end{aligned}
$$

Proof. One first checks that the general fibre is a smooth $K3$-surface. For this it suffices to look at the hypersurface in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

In the particular example the total space has at the origin of the affine chart $x_1 = 1$ a singularity, which is isomorphic to the cone over the Del Pezzo surface of degree 5, as it should be: the point to check is that we indeed have a generic local deformation. Furthermore there are 18 singularities of type A_1. On the (x_1, x_4)-line we have the point $x_1 + x_4 = 0$. On the (x_0, x_1)-line we have two points, given by $x_0^2 + x_0x_1 + x_1^2$, and on the (x_7, x_7)-line the two points $x_6^2 + x_6x_7 + x_7^2$. The other singular points are found by symmetry.

By blowing up the three singularities of multiplicity 5 and making a small resolution of the A_1-points we get a smooth total space. To obtain the (-1)-form one has to place one exceptional curve on either component in case the double line contains two singularities. If there is only one, the exceptional curve should lie on the triangle component. □

REFERENCES

[A'C] A'CAMPO, N. La fonction zêta d'une monodromie. *Comment. Math. Helv. 50* (1975), 233–248.

[A-C] ALTMANN, K. and J. A. CHRISTOPHERSEN. Deforming Stanley-Reisner rings. arXiv:math.AG/0006139

[B-S] BAYER, D. and M. STILLMAN. Macaulay: A system for computation in algebraic geometry and commutative algebra. See http://www.math.columbia.edu/~bayer/Macaulay/

[D-E-P] DE CONCINI, C., D. EISENBUD and C. PROCESI. Hodge algebras. *Astérisque 91* (1982).

[F1] FRIEDMAN, R. Base change, automorphisms, and stable reduction for type III $K3$ surfaces. In: [SAGS], pp. 277–298.

[F2] —— Global smoothings of varieties with normal crossings. *Ann. of Math. (2) 118* (1983), 75–114.

[F-M] FRIEDMAN, R. and D. R. MORRISON. The birational geometry of degenerations: an overview. In: [SAGS], pp. 1–32.

[F-S] FRIEDMAN, R. and F. SCATTONE. Type III degenerations of $K3$ surfaces. *Invent. Math. 83* (1985), 1–39.

[K] KULIKOV, V. S. On modifications of degenerations of surfaces with $\kappa = 0$. *Izv. Akad. Nauk SSSR, Ser. Mat. 44* (1980), 1115–1119 = *Math. USSR, Izv. 17* (1981), 339–342.

[L] LOOIJENGA, E. Rational surfaces with an anticanonical cycle. *Ann. of Math. (2) 114* (1981), 267–322.

[M-M] MIRANDA, R. and D. R. MORRISON. The minus one theorem. In: [SAGS], pp. 173–259.

[Mu] MUKAI, S. Finite groups of automorphisms of $K3$ surfaces and the Mathieu group. *Invent. Math. 94* (1988), 183–221.

[N] NISHIGUCHI, K. Degeneration of $K3$ surfaces. *J. Math. Kyoto Univ. 28* (1988), 267–300.

[P] PERSSON, U. On degenerations of algebraic surfaces. *Mem. Amer. Math. Soc. 11* (1977), no. 189.

[S1] STEVENS, J. Computing versal deformations. *Experiment. Math. 4* (1995), 129-144.

[S2] —— Deformations of cones over hyperelliptic curves. *J. Reine Angew. Math. 473* (1996), 87–120.

[SAGS] *The Birational Geometry of Degenerations.* Editors: Robert Friedman and David R. Morrison. Birkhäuser, Boston, 1983 (Progress in Mathematics, 29).