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which is always positive since the matrix $(H\overline{H}+\overline{H}H)$ is positive semi-definite, so its trace is ≥ 0. Hence the maps α and Φ are well defined. It is clear that the image of Φ is contained in $S^4 \subset \mathcal{S}$, because the linearity of the trace implies that

$$[\mathrm{trace}(\Phi(H))]^2 = \alpha^2(H)\,[\mathrm{trace}\,\psi(H)]^2 = 1\,.$$

It is also clear that Φ is $\mathrm{SO}(3,\mathbf{R})$-equivariant, since the trace is invariant under conjugation and ψ is equivariant by Lemma 3.7. These considerations imply both Lemma 3.8 and the following

LEMMA 3.9. *The map Φ is an equivariant surjection from $P(2)$ over $S^4 \subset \mathcal{S}$, and it is two-to-one, except over the image of the real matrices in $P(2)$ where it is one-to-one.*

This gives the map in Theorem 3.4 that determines an equivariant diffeomorphism between S^4 and $P^2_{\mathbf{C}}$ modulo the involution given by conjugation. To complete the proof of Theorem 3.4 we need to show that Φ is invariant under the involution of $P(2)$ that corresponds to complex conjugation in $P^2_{\mathbf{C}}$. For this we notice that if L_H is the complex line in $\mathbf{C}^3$ which is the image of $H \in P(2)$, and if $0 \neq (z_1, z_2, z_3) \in L_H$, we can associate to H the point in $P^2_{\mathbf{C}}$ with projective coordinates $[z_1, z_2, z_3]$. To the matrix $\overline{H}$ there corresponds the line with projective coordinates $[\overline{z}_1, \overline{z}_2, \overline{z}_3]$. Therefore we have

LEMMA 3.10. *The involution $j*$ of $P(2)$ defined by $j*(H) = \overline{H}$ coincides with the involution j of $P^2_{\mathbf{C}}$ given by complex conjugation, $[z_1, z_2, z_3] \overset{j}{\mapsto} [\overline{z}_1, \overline{z}_2, \overline{z}_3]$.*

Then Φ is invariant under this involution, since $\Re(H) = \Re(\overline{H})$, proving Theorem 3.4. □

4. SOME APPLICATIONS AND REMARKS

It is interesting to describe explicitly the orbits of the Γ action of $\mathrm{SO}(3,\mathbf{R})$ on S^4, regarded[2]) as the set of matrices with norm 1 in $\mathcal{S}$. In fact, the orbits of this action are conjugacy classes (or *congruency classes*) of traceless symmetric matrices whose square has trace 1. This is the connection between our construction and the spherical Tits buildings. Every $S \in \mathcal{S}$ can

[2]) This orbit description of S^4 is also given in [Ma2].

be diagonalized by an element in $SO(3,\mathbf{R})$, hence every orbit has a unique representative which is diagonal. So let us assume that S is diagonal with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. The two special orbits correspond to the cases when two eigenvalues coincide. Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, if two eigenvalues coincide we must have, up to conjugation, $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{6}}$ and $\lambda_3 = -\frac{2}{\sqrt{6}}$, or $\lambda_1 = \lambda_2 = -\frac{1}{\sqrt{6}}$ and $\lambda_3 = \frac{2}{\sqrt{6}}$. In both cases the corresponding matrix is determined by the plane P given by the two equal eigenvalues, say λ_1 and λ_2. Equivalently, this matrix is determined by the line orthogonal to P, in which we act by the multiplier $\lambda_3 = \pm\frac{2}{\sqrt{6}}$; the sign here distinguishes the two orbits. Since $SO(3,\mathbf{R})$ acts transitively on the lines in $\mathbf{R}^3$, it follows that each of these special orbits is a copy of $P^2_{\mathbf{R}}$, as we know from [HL]. The general orbits occur when the three eigenvalues are distinct and the corresponding eigenspaces are orthogonal lines. Since the trace is 0, two eigenvalues determine the third. Hence in each case the transformation is determined by the plane P given by two eigenvalues and the line l in P given by one of them, together with the corresponding multipliers on l, on the line orthogonal to l in P and on the line orthogonal to P in $\mathbf{R}^3$. That is, we have a flag (P,l) in $\mathbf{R}^3$, together with the multipliers λ_1, λ_2 and λ_3. Since the action of $SO(3,\mathbf{R})$ is transitive on the planes in $\mathbf{R}^3$ and on the lines in each such plane, it follows that each principal orbit, a copy of the flag manifold $F^3(2,1)$, is the orbit of the flag (P,l). The different orbits correspond to the different multipliers.

We also notice that there is a double fibration, similar to the one considered in (1.4) above:

(4.1)
$$\begin{array}{ccc} & F^3(2,1) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ P^2_{\mathbf{R}} & & P^2_{\mathbf{R}} \end{array} .$$

where $\pi_1(P,l) = l$ and $\pi_2(P,l) = P$. We can form the corresponding double mapping cylinder $\big(F^3(2,1) \times [0,1]\big)/\!\sim$, where $\sim$ identifies a point $((P_0,l_0),0) \in \big(F^3(2,1) \times \{0\}\big)$ with the point $\pi_1(P_0,l_0) = l_0 \in P^2_{\mathbf{R}}$, and a point $((P_1,l_1),1) \in \big(F^3(2,1) \times \{1\}\big)$ with the point $\pi_2(P_1,l_1) = P_1 \in P^2_{\mathbf{R}}$. We obtain S^4.

The double fibration given by (1.4) in this dimension descends to (4.1) by conjugation. By the previous discussion, the image of Q in S^4 is the copy of $P^2_{\mathbf{R}}$ which is the orbit of the diagonal matrix with eigenvalues $\left\{-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\}$, while Π is taken diffeomorphically into the orbit of $\left\{\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right\}$.

Since the action of $SO(3, \mathbf{R})$ in S^4 is by isometries and is transitive on each orbit, the principal orbits are at constant distance from each of these exceptional orbits $M_1 \cong P^2_{\mathbf{R}}$ and $M_2 \cong P^2_{\mathbf{R}}$, i.e. they are "parallel". In other words, as in Section 2, the principal orbits are the level sets of the function $f\colon S^4 \to \mathbf{R}$ given by $f(x) = (d(x, M_1))^2$ (or the level sets of the function $g(x) = (d(x, M_2))^2$). Both f and g are smooth Bott-Morse functions (cf. [DR]).

The fixed-point free involution on S^4 given by $\iota\colon A \in \mathcal{S} \mapsto -A \in \mathcal{S}$ commutes with our $SO(3, \mathbf{R})$ action and therefore it takes $SO(3, \mathbf{R})$-orbits into orbits. The quotient S^4/ι is the real projective space $P^4_{\mathbf{R}}$, equipped with an isometric $SO(3, \mathbf{R})$-action. The two exceptional orbits M_1 and M_2 on S^4 are identified by ι. Thus we have only one exceptional orbit for the action of $SO(3, \mathbf{R})$ on $P^4_{\mathbf{R}}$. The orbit N of the matrix in S^4 which corresponds to the matrix in $\mathcal{S}$ whose eigenvalues are $\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$ is the manifold consisting of points such that $d(x, M_1) = d(x, M_2)$. Then, N is invariant under ι and separates S^4 into two regions which are interchanged by ι (i.e. N is an "equator" for the orientation-reversing involution ι). The orientable 3-manifold N is the flag manifold described earlier, but it can also be described as the set of ordered pairs (l_1, l_2) of *non-oriented* lines of $\mathbf{R}^3$ which are mutually orthogonal. These lines are the eigenspaces corresponding to the eigenvalues $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$, respectively.

The restriction of ι to N is the orientation-preserving and fixed-point free involution given by $(l_1, l_2) \mapsto (l_2, l_1)$. Let π denote the double covering map from S^4 to $P^4_{\mathbf{R}} = S^4/\iota$. Let $\pi(M_1) = \pi(M_2) := M \cong P^2_{\mathbf{R}}$ and $\pi(N) := \widehat{N}$. The manifold $\widehat{N}$ is diffeomorphic to $SO(3, \mathbf{R})/D_4$, where D_4 is the group of order 8 of isometries of the square. This is because $SO(3, \mathbf{R})$ acts transitively on the set of non-oriented pairs $\{l_1, l_2\}$ of lines in $\mathbf{R}^3$ which are mutually orthogonal and the isotropy group is precisely D_4. Therefore $\widehat{N}$ is diffeomorphic to $SU(2)/\widetilde{D}_4 \cong S^3/\widetilde{D}_4$, where $\widetilde{D}_4$ is the binary dihedral group of order 16, i.e. $\widetilde{D}_4 = \phi^{-1}(D_4)$ where $\phi\colon S^3 \cong SU(2) \to SO(3, \mathbf{R})$ is the canonical epimorphism.

The embedding $P^2_{\mathbf{R}} \cong M \subset P^4_{\mathbf{R}}$ is exactly the embedding given by the Veronese embedding $P^2_{\mathbf{R}} \to S^4$, followed by the canonical projection from S^4 into $P^4_{\mathbf{R}}$ (see [HL]). We know that $S^4 \setminus (M_1 \cup M_2)$ is diffeomorphic to $N \times \mathbf{R}$, and the restriction of the involution ι to $S^4 \setminus (M_1 \cup M_2)$ is conjugate to the involution $\mathfrak{I}$ of $N \times \mathbf{R}$ given by $((l_1, l_2), t) \mapsto ((l_2, l_1), -t)$. Therefore the quotient $(N \times \mathbf{R})/\mathfrak{I}$ is diffeomorphic to the total space of the non-orientable line bundle over $\widehat{N}$. Summarizing, we have the following

COROLLARY 4.2. *Let $P^2_{\mathbf{R}} \cong M \subset P^4_{\mathbf{R}}$ be the embedding induced by the classical Veronese embedding $P^2_{\mathbf{R}} \hookrightarrow S^4$. Then $P^4_{\mathbf{R}} \setminus M$ is diffeomorphic to the total space of the non-orientable real line bundle over* $\mathrm{SU}(2)/\widetilde{D}_4 = S^3/\widetilde{D}_4$. *In particular the fundamental group of $P^4_{\mathbf{R}} \setminus M$ is the binary dihedral group of order* 16.

Let us now recall that there is a remarkable fibre bundle $\pi \colon P^3_{\mathbf{C}} \to S^4$ with fibre $P^1_{\mathbf{C}}$, called *the twistor fibration*, or also the *Calabi-Penrose fibration* (we refer to [Sa, SV] for details). The fibres are called *the twistor lines*. There are several equivalent ways to construct this fibration. The standard way is to think of $P^3_{\mathbf{C}}$ as being the homogeneous space $\mathrm{SO}(5,\mathbf{R})/U(2)$, which fibres over $\mathrm{SO}(5,\mathbf{R})/\mathrm{SO}(4,\mathbf{R}) \cong S^4$ with fibre $\mathrm{SO}(4,\mathbf{R})/U(2) \cong S^4 \cong P^1_{\mathbf{C}}$. A more geometric way of describing this twistor fibration is to consider S^4 as being the quaternionic projective line $P^1_{\mathcal{H}}$, of *right* quaternionic lines in the quaternionic plane $\mathcal{H}^2$ (regarded as a 2-dimensional *right* $\mathcal{H}$-module). That is, for $q := (q_1, q_2) \in \mathcal{H}^2$ $(q \neq (0,0))$, the right quaternionic line passing through q is the linear space

$$R_q := \{(q_1\lambda, q_2\lambda) \mid \lambda \in \mathcal{H}\}.$$

We can identify $\mathcal{H}^2$ with $\mathbf{C}^4$ via the $\mathbf{R}$-linear map given by $(q_1, q_2) \mapsto (z_1, z_2, z_3, z_4)$, where $q_1 = z_1 + z_2\mathbf{j} = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ and $q_2 = z_3 + z_4\mathbf{j} = y_1 + y_2\mathbf{i} + y_3\mathbf{j} + y_4\mathbf{k}$. In this notation $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the standard quaternionic units, $z_1 = x_1 + x_2\mathbf{i}$, $z_2 = x_3 + x_4\mathbf{i}$, $z_3 = y_1 + y_2\mathbf{i}$ and $z_4 = y_3 + y_4\mathbf{i}$.

Under this identification each right quaternionic line is invariant under *right* multiplication by $\mathbf{i}$. Hence such a line is canonically isomorphic to $\mathbf{C}^2$. If we think of $P^3_{\mathbf{C}}$ as being the space of complex lines in $\mathbf{C}^4$, then there is an obvious map $\pi \colon P^3_{\mathbf{C}} \to S^4$, whose fibre over a point $H \in P^1_{\mathcal{H}}$ is the space of complex lines in the given right quaternionic line $H \cong \mathbf{C}^2$; thus the fibre is $P^1_{\mathbf{C}}$.

The group $\mathrm{Conf}_+(S^4)$ of orientation preserving conformal automorphisms of S^4 is isomorphic to $P\mathrm{SL}(2,\mathcal{H})$, the projectivization of the group of 2×2, invertible, quaternionic matrices. This is naturally a subgroup of $P\mathrm{SL}(4,\mathbf{C})$, since every quaternion corresponds to a couple of complex numbers. Hence $\mathrm{Conf}_+(S^4)$ has a canonical lifting to a group of holomorphic transformations of $P^3_{\mathbf{C}}$, carrying twistor lines into twistor lines.

Let us split (differentiably) the tangent bundle of $P^3_{\mathbf{C}}$ into a horizontal sub-bundle and a "vertical" sub-bundle (the bundle tangent to the twistor fibres), via the Levi-Civita connexion of the metric. Since the lifting of $\mathrm{Conf}_+(S^4)$ permutes the twistor lines, this action on $TP^3_{\mathbf{C}}$ preserves the decomposition into horizontal and vertical sub-bundles. By [SV], the action on the vertical sub-bundle is by isometries with respect to the Fubini-Study metric (which is just the standard metric on S^2). We remark that the horizontal bundle is a *holomorphic* complex sub-bundle of rank two of the complex tangent bundle of $P^3_{\mathbf{C}}$. On this sub-bundle, the action is conformal. However, the group $\mathrm{SO}(5,\mathbf{R})$ is a subgroup of $\mathrm{Conf}_+(S^4)$ and, by construction, its induced action on the horizontal sub-bundle is by isometries. Thus we have an isometric action of $\mathrm{SO}(5,\mathbf{R})$ on $P^3_{\mathbf{C}}$, with respect to the Fubini-Study metric, which restricts to an isometric action of $\mathrm{SO}(3,\mathbf{R})$ on $P^3_{\mathbf{C}}$, via the representation Γ of this group in $\mathrm{SO}(5,\mathbf{R})$ discussed earlier. We denote this latter action of $\mathrm{SO}(3,\mathbf{R})$ on $P^3_{\mathbf{C}}$ by $\check{\Gamma}$.

We notice that the special orbits of the $\mathrm{SO}(3,\mathbf{R})$-action on S^4 give rise to the special orbits in $P^3_{\mathbf{C}}$, each being diffeomorphic to $P^2_{\mathbf{R}}$. There is one such orbit for each point in the twistor line over a point in the corresponding special orbit in S^4. Since the twistor bundle is trivial when restricted to any proper subset of S^4, it follows that the set of all special orbits of each type is diffeomorphic to $P^2_{\mathbf{R}} \times P^1_{\mathbf{C}}$. Similar remarks apply to the principal orbits. Moreover, by [HL], each special orbit is a minimal submanifold of $P^3_{\mathbf{C}}$, and so is their product $P^2_{\mathbf{R}} \times P^1_{\mathbf{C}}$ since the projection $P^3_{\mathbf{C}} \to S^4$ is a harmonic map which is a Riemannian fibration (i.e. it is transversally isometric), by [EL] and [EV; 7.9]. Thus we have

THEOREM 4.3. *The action* $\check{\Gamma}$ *of* $\mathrm{SO}(3,\mathbf{R})$ *on* $P^3_{\mathbf{C}}$ *is such that:*

(1) *The action is by elements of* $P\mathrm{SU}(4)$, *i.e. by isometries of* $P^3_{\mathbf{C}}$ *that permute the twistor lines, sending each twistor line isometrically onto its image.*

(2) *There are two exceptional types of orbits, each of which is diffeomorphic to* $P^2_{\mathbf{R}}$. *If we denote by* K_1 *and* K_2 *the union of orbits of each of these two types, then both* K_1 *and* K_2 *are diffeomorphic to* $P^2_{\mathbf{R}} \times P^1_{\mathbf{C}}$. *Furthermore,* K_1 *and* K_2 *are minimally embedded in* $P^3_{\mathbf{C}}$.

(3) *The principal orbits are diffeomorphic to* $F^3(2,1)$. *Hence the action has cohomogeneity* 3.

(4) *The functions* $h_1\colon P^3_{\mathbf{C}} \to \mathbf{R}$ *and* $h_2\colon P^3_{\mathbf{C}} \to \mathbf{R}$, *given by* $h_1(Z) = (d(Z,K_1))^2$ *and* $h_2(Z) = (d(Z,K_2))^2$, *are both Bott-Morse functions with critical set* $K_1 \cup K_2$.

(5) *The space of orbits is* $S^2 \times [0,1]$.

We may now consider the Hopf fibration $\tilde{\pi}\colon S^7 \to S^4$ and we identify $\mathbf{R}^8 \cong \mathcal{H}^2$ in the obvious way.

The group SU(2) consists of all 2×2 complex matrices of the form $\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ with determinant 1. This group can be identified with the group Sp(1) of unit quaternions by mapping each such matrix to the unit quaternion $u = z_1 + z_2\mathbf{j}$. Hence SU(2) acts by the right on $\mathcal{H}^2 \cong \mathbf{R}^8$ by the map $((q_1, q_2), u) \mapsto (q_1 u, q_2 u)$, for each $u \in \mathrm{Sp}(1)$ and $(q_1, q_2) \in \mathcal{H}^2$. This action leaves invariant each right line R_q $(q = (q_1, q_2))$ and it acts as an isometry on this line.

On the other hand, each complex number is a quaternion, so each matrix in SU(2) can be regarded as a 2×2 quaternionic matrix in $\mathrm{GL}(2, \mathcal{H})$, the group of all invertible 2×2 quaternionic matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This group acts on $\mathcal{H}^2$ by the left according to the formula

$$q = (q_1, q_2) \mapsto (aq_1 + bq_2, cq_1 + dq_2) = A(q)\,,$$

and induces the aforementioned action of $P\mathrm{SL}(2, \mathcal{H})$ on $P^1_{\mathcal{H}} \cong S^4$.

We thus have an action of $\mathrm{SU}(2) \times \mathrm{Sp}(1)$ on $\mathbf{R}^8 \cong \mathcal{H}^2$ by the formula

$$((g, u), (q_1, q_2)) \mapsto (a_g q_1 u + b_g q_2 u, c_g q_1 u + d_g q_2 u)$$

for each $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$ in SU(2). This action induces a natural action

$$\widehat{\Gamma}\colon (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times S^7 \to S^7$$

on the sphere S^7, and this action is a lifting of the action Γ considered in Section 3, i.e. the following diagram is commutative:

$$\begin{array}{ccc} (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times S^7 & \xrightarrow{\widehat{\Gamma}} & S^7 \\ {\scriptstyle f \times \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ \mathrm{SO}(3, \mathbf{R}) \times S^4 & \xrightarrow{\Gamma} & S^4\,, \end{array}$$

where $f(g, u) = \phi(g)$, ϕ being the canonical epimorphism from SU(2) to $\mathrm{SO}(3, \mathbf{R})$). It is clear that $\widehat{\Gamma}((-Id, -1), x) = x$ for all $x \in S^7$, so $\widehat{\Gamma}$ actually descends to an action of

$$\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{Sp}(1)/(\mathbf{Z}/2\mathbf{Z})\,,$$

on S^7. We have

THEOREM 4.4. *This* SO(4) *action on* S^7 *satisfies:*

(1) *It is a hyperpolar isometric action of cohomogeneity 1, with space of orbits the interval* $[0, \frac{\pi}{2}]$.

(2) *The two exceptional orbits are both diffeomorphic to* $P^2_{\mathbf{R}} \times S^3$ *and both are minimally embedded in* S^7.

(3) *The principal orbits are diffeomorphic to* $F^3(2,1) \times S^3$.

(4) *The square of the distance functions to the exceptional orbits are both Bott-Morse functions.*

(5) *The union of the two exceptional orbits, both copies of* $P^2_{\mathbf{R}} \times S^3$, *is the Spanier-Whitehead dual of one principal orbit* $F^3(2,1) \times S^3$.

We notice that the action of $\mathrm{SO}(n+1)$ on $\mathbf{C}^{n+1}$ considered in Section 2 also provides, when $n = 3$, an isometric action of cohomogeneity 1 of SO(4) on S^7. However, in this case the two special orbits are the inverse images of the quadric Q and the real projective space $\Pi \cong P^3_{\mathbf{R}}$ under the projection $S^7 \to P^3_{\mathbf{C}}$. So this action is not equivalent to the "twistorial" one given by Theorem 4.4.

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