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degree by a kind of geometric complexity (in spirit of Thom) of a smooth map, making use of a quantitative form of the Thom transversality theorem instead of Bézout's theorem. The quantitative transversality can be used also for counting periodic orbits of a vector field and periodic points of a transformation preserving an additional structure (volume, symplectic form, etc.) and it yields the Artin-Mazur estimate for dense sets of such maps. Unfortunately, the detailed proof (at least the one I know of) is more lengthy than the algebraic one, and I shall treat the subject somewhere else.

REMARK. The quantitative transversality theory has been developed by Y. Yomdin (see p. 124 in [5′] for a brief introduction) but does not suffice, as it stands, for the diff-version of the Artin-Mazur theorem. In fact, one needs here an adequate notion of genericity (compare remark on p. 31 in [5′]) as is shown in [1′] for smooth maps. I have never returned to this issue and can only conjecture the extension (and sharpening) of the corresponding results in [1′] to structure preserving maps and/or vector fields.

§5. QUASICONFORMAL MAPS

For a smooth map $f: X \to Y$ from one oriented n-dimensional Riemannian manifold into another, we denote by $D_x f$ its differential at x, by $\|D_x f\|$ the norm of the differential, by $\det D_x f$ its Jacobian, and by $\lambda_x f$ the ratio $\|D_x f\|^n / \det D_x f$ called the *conformal dilation* at $x \in X$. A map is called λ-quasiconformal if, for almost all x, the differential $D_x f$ exists, $\det D_x f$ does not vanish, and $\lambda_x f \leq \lambda$. A quasiconformal map must have locally positive degree. If $n = 1$, each locally diffeomorphic map is conformal (i.e. 1-quasiconformal).

When $n = 2$, conformal maps are complex analytic and for $n > 2$ all conformal maps are locally diffeomorphic. In particular, when $n > 2$, every non-injective conformal endomorphism is conjugate to a homothety of a flat Riemannian manifold. When $\lambda > 1$, there are (not locally injective) λ-quasiconformal maps in all dimensions $n > 1$. They are locally homeomorphic outside a codimension 2 branching set. At that set, they are never $(n > 2)$ smooth.

GENERALISATION OF (0.1)

If f is a λ-quasiconformal endomorphism of a closed n-dimensional manifold, then

$$h(f) \leq \log \deg f + n \log \lambda . \tag{5.0}$$

(Compare with the standard estimate $h(f) \leq \sup_{x \in X} n \log \|D_x f\|$.)

Proof. As before, we estimate the density and volume of the iterated graph and we need an analogue of (2.0) only for the Euclidean space. The only new ingredient here is the following obvious fact:

Consider an n-dimensional $V \subset (\mathbf{R}^n)^k$ with all projections $V \to \mathbf{R}^n$ $\tilde{\lambda}$-quasiconformal and having volumes not greater than $\mu > 0$. (The volume of a map is the integral of its Jacobian.) Then

$$\text{Vol } V \leq k^{n+1} \tilde{\lambda} \mu . \tag{5.1}$$

Combining (5.1) with the isoperimetric inequality applied to the projection $V \to \mathbf{R}^n$ realizing μ we conclude that

$$\text{Vol } V \leq C k^{n+1} \tilde{\lambda} A^{\frac{n}{n-1}} , \qquad C = C(n) , \tag{5.2}$$

where A denotes the $(n-1)$-dimensional volume of the boundary ∂V. (In other words, graphs of quasiconformal maps are quasiminimal.)

Now, using the same notation as in Section 2, we conclude that

$$V_\epsilon \leq C k^{n+1} \tilde{\lambda} A_\epsilon^{\frac{n}{n+1}} \tag{5.3}$$

and combining (5.3) with (2.2) and (2.3), we have:

$$V_\epsilon \geq \text{const}_n K^{\text{constant}_n} n \epsilon^n \tilde{\lambda}^{1-n} . \tag{5.4}$$

Combining (5.4) and (1.1) and observing that projections of the iterated graph $(\Gamma_f)_k$ of a λ-quasiconformal f are λ^k-quasiconformal we obtain:

$$h(f) \leq \text{lov } \Gamma_f + (n-1) \log \lambda .$$

To complete the proof, we apply (5.1) again and get

$$\text{lov } \Gamma_f \leq \log \deg f + \log \lambda .$$