1. An illustration

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

1. An illustration

We start by considering a very particular case of our theorem. We feel that this simple example might serve as an illustration of the later work. We hope that this will help the reader to understand the contents and ideas of the paper. Our aim is to prove the Affirmation stated below.

We choose a real number $\lambda > 1$. We denote by d_0 the usual distance on $\mathbf{R}$. For any real r, we set $d_r = \lambda^{|r|} d_0$. The length $|I|_r$ of a real interval I is the distance, with respect to d_r, between the endpoints of I. We consider the plane $\mathbf{R}^2$. We denote by $p_x \colon \mathbf{R}^2 \to \mathbf{R}$ the projection on the x-axis and by $p_y \colon \mathbf{R}^2 \to \mathbf{R}$ the projection on the y-axis. We denote by $V_a = p_x^{-1}(a)$ the vertical line through a point a. Vertical lines (resp. horizontal lines $p_y^{-1}(r)$) are equipped with the distance d_0 (resp. with the distance d_r). Lengths of horizontal and vertical intervals are measured with respect to the distance defined on the corresponding line. A *telescopic path* is a concatenation of non degenerate vertical and horizontal intervals, where 'non degenerate' means not reduced to a point. The horizontal (resp. vertical) length of a telescopic path is the sum of the horizontal (resp.vertical) lengths of its maximal horizontal (resp. vertical) intervals. The telescopic length of a telescopic path is the sum of its horizontal and vertical lengths. The telescopic distance between two points in $\mathbf{R}^2$ is the infimum of the telescopic lengths of the telescopic paths between these two points. We wish to prove the following result:

AFFIRMATION. *The plane $\mathbf{R}^2$ equipped with the telescopic distance is a Gromov hyperbolic geodesic metric space.*

STEP 1: COMPUTATION OF THE GEODESICS. Let a, b be any two points in $\mathbf{R}^2$. Let I_{ab} be the compact interval of the x-axis bounded by the projections $p_x(a)$ and $p_x(b)$ of a and b. Let g be any telescopic geodesic from a to b. On the one hand, the length of a telescopic path is never shorter than the length of its projection on a vertical line, so that g lies between V_a and V_b. On the other hand, if $c \in I_{ab}$, the vertical line V_c separates a from b, so that g intersects V_c. *Therefore the telescopic geodesic g intersects all the vertical lines separating a from b, and no other vertical line.* Given a telescopic path containing one vertical interval and two horizontal intervals I, I' at different heights, there exists a stricly shorter telescopic path with the same endpoints. It is obtained by replacing one of the horizontal intervals, say I, by another horizontal interval which intersects the same vertical lines as I, and which lies at the same height as I'. *Thus the telescopic geodesic* g *is the concatenation of at most one non degenerate horizontal interval with at most two non degenerate vertical intervals.* Furthermore, any horizontal interval on the x-axis minimizes the horizontal distance between the vertical lines passing through its endpoints. *Thus, if* $p_y(a)p_y(b) \leq 0$ *then* g *is the concatenation of the horizontal interval* I *on the* x*-axis which connects* V_a *and* V_b*, with the vertical intervals on* V_a *and* V_b *which connect* a *and* b *to the endpoints of* I.

In order to compute the geodesics when $p_y(a)p_y(b) \geq 0$, we distinguish two cases:

CASE A: $0 \leq p_y(a) = p_y(b)$. Then g is the concatenation of two vertical intervals of vertical lengths $t \geq 0$ with one horizontal interval I. The horizontal length of I is equal to $\lambda^t d_{p_y(a)}(a,b)$ if $p_y(I) \geq p_y(a)$ and to $\lambda^{-t} d_{p_y(a)}(a,b)$ if $p_y(I) \leq p_y(a)$ and $p_y(I) \geq 0$. Indeed, we recall that horizontal intervals on the x-axis are dilated both in the future and in the past. We set $f(t) = 2t + \lambda^{-t} d_{p_y(a)}(a,b)$. Let $t_\star$ be any real number such that $0 \leq t_\star \leq p_y(b)$ and $f(t_\star) = \min_{0 \leq t \leq p_y(b)} f(t)$. From what precedes, g is the concatenation of two vertical intervals of length $t_\star$ with a horizontal interval on the horizontal line $p_y^{-1}(p_y(b) - t_\star)$. The function $f(t)$ attains its minimum at $t_\circ = \frac{\ln((\ln \lambda) d_{p_y(a)}(a,b)/2)}{\ln \lambda}$. Therefore $t_\star = \min(\max(t_\circ, 0), p_y(b))$ is unique. *We have thus proved that there exists a unique telescopic geodesic between* a *and* b. Its telescopic length is equal to $f(t_\star)$.

We now distinguish three subcases.

Case (0): $t_\star > t_\circ$. *The horizontal distance between* a *and* b *is so short that the horizontal interval between* a *and* b *realizes the telescopic distance.* Indeed $t_\star > t_\circ \Rightarrow t_\star = 0$. The horizontal distance between a and b, which is the horizontal length of the horizontal interval I in the above notation, is smaller than $\frac{2}{\ln \lambda}$.

Case (1): $t_\star = t_\circ$. *The optimal case.* The horizontal interval I of g lies on the horizontal line $p_y(a) - t_\circ$. The horizontal length of I is $\frac{2}{\ln \lambda}$. The vertical intervals in g have vertical lengths $t_\circ$.

Case (2): $t_\star < t_\circ$. *The horizontal distance between* a *and* b *is too large with respect to the height of the horizontal line through* a *and* b. Then the horizontal interval I of g lies on the x-axis. The horizontal length of I is equal to $\lambda^{-p_y(a)} d_{p_y(a)}(a,b) > \frac{2}{\ln \lambda}$. It depends on $d_{p_y(a)}(a,b)$ and can be arbitrarily large.

CASE B: $0 \leq p_y(a) \neq p_y(b)$. Without loss of generality we assume that $p_y(a) < p_y(b)$. We consider the point $c = V_a \cap p_y^{-1}(p_y(b))$. If $t_\star \geq p_y(b) - p_y(a)$, the telescopic geodesic from c to b computed in Case A admits a subpath from a to b. This subpath is the unique telescopic geodesic between a and b. If $t_\star < p_y(b) - p_y(a)$, then the unique telescopic geodesic between a and b is the concatenation of the horizontal interval between a and the vertical through b, with the vertical segment between this interval and the point b.

The same arguments apply to the case where both a and b lie in the negative half-plane. This concludes the computations of the geodesics.

STEP 2: GEODESIC TRIANGLES ARE THIN. Let Δ be any geodesic triangle in the upper half-plane. Let g_1, g_2, g_3 be the sides of Δ. Let $t_\star(g_i)$ and $t_\circ(g_i)$ be the non negative real numbers for g_i defined above. Let I_1, I_2, I_3, $p_y(I_3) \geq p_y(I_2) \geq p_y(I_1)$, be the horizontal geodesics respectively in g_1, g_2 and g_3.

Case (1): $t_\star(g_1) \geq t_\circ(g_1)$. Then $t_\star(g_2) \geq t_\circ(g_2)$ and $t_\star(g_3) \geq t_\circ(g_3)$. Therefore $|I_i|_{p_y(I_i)} \leq \frac{2}{\ln \lambda}$, $i = 1, 2, 3$. The vertical segment of g_2 between I_3 and I_2 is at horizontal distance smaller than $\frac{2}{\ln \lambda}$ from a vertical segment in g_1. Because of the uniform contraction in λ^{-t}, this implies that I_2 is at vertical distance smaller than $\frac{\ln 2}{\ln \lambda}$ from I_1. Therefore the union of I_1 with the two orbit-segments between its endpoints and the horizontal line $p_y^{-1}(p_y(I_2))$ is at telescopic distance smaller than $\frac{\ln 2}{\ln \lambda} + \frac{2}{\ln \lambda}$ from I_2. All the points of Δ not considered up to now belong to at least two distinct sides.

Case (2): $t_\star(g_1) < t_\circ(g_1)$. Then $p_y(I_1) = 0$, i.e. I_1 lies on the x-axis.

1. If $t_\star(g_2) = t_\circ(g_2)$ and $t_\star(g_3) = t_\circ(g_3)$, then $|I_i|_{p_y(I_i)} = \frac{2}{\ln \lambda}$ for $i = 2, 3$. Thus $|I_1|_0 \leq \frac{4}{\ln \lambda}$. We conclude as in Case (1).

2. If both $t_\star(g_2) > t_\circ(g_2)$ and $t_\star(g_3) > t_\circ(g_3)$ then both I_2 and I_3 lie on the x-axis so that $I_1 = I_2 \cup I_3$. Then any point in Δ belongs to at least two distinct sides.

3. If only $t_\star(g_3) > t_\circ(g_3)$ then $I_2 \subset I_1$. Let $I_1' \subset I_1$ be the complement of I_2 in I_1. Then $|I_1'|_0 \leq \frac{2}{\ln \lambda}$. The same inequality is satisfied for the horizontal distance between the vertical segments connecting the endpoints of I_1' to I_3. This concludes Case (2).

The case where Δ lies in the negative half-plane is treated in the same way. The other cases are dealt with using similar, but simpler, arguments than above. We leave them as an exercise for the reader.

REMARK 1.1. The above computations fail, and the space is no longer Gromov-hyperbolic, if one replaces $d_y = \lambda^{|y|} d_0$ by $d_y = P(|y|) d_0$, where $P(\,.\,)$ is a polynomial function of y. Indeed, in this case, the length of the horizontal interval between the two considered orbits, evaluated at the height where the minimum of the length-function $f(t)$ is attained, depends, even in the optimal case, on the horizontal length of the interval connecting one point to the orbit of the other. Whereas in the exponential case it equals $\frac{2}{\ln \lambda}$ unless it belongs to the horizontal axis.

2. MAPPING-TELESCOPES AND FOREST-STACKS

Let X be a topological space. Call X a *topological tree* if there exists a unique arc between any two points in X. A *topological forest* is a union of disjoint topological trees. By 'arc' we mean the image of an injective path. A path in X is a continuous map from a bounded interval of the real line into X. A *forest-map* is a continuous map of a topological forest into itself.

DEFINITION 2.1. Let $\psi\colon X \to X$ be a forest-map. The *mapping-telescope* K_ψ *of* (ψ, X) is the topological space resulting from $K_X = \bigsqcup_{n \in \mathbf{Z}} X \times [n, n+1]$ by the identification of each point $(x, n+1) \in X \times [n, n+1]$ with the point $(\psi(x), n+1) \in X \times [n+1, n+2]$.

Let us examine somewhat more closely the topology of these mapping-telescopes.

For any integer $n \in \mathbf{Z}$, for any $(x, r) \in X \times [n, n+1]$, for any real number $t \geq 0$, we define $\tilde{\sigma}_t((x, r))$ as the point $(\psi^{E[t-(n+1-r)]+1}(x), r+t)$ in $X \times [E[r+t], E[r+t]+1]$, where $E[r]$ denotes the integer part of r. The map $\tilde{\sigma}_t$ is defined on K_X (the disjoint union of the $X \times [n, n+1]$) for every $t \geq 0$. Moreover $\tilde{\sigma}_{t+t'} = \tilde{\sigma}_t \circ \tilde{\sigma}_{t'}$.

If $a = (x, n+1) \in X \times [n+1, n+2]$, then $\tilde{\sigma}_t(a) = (\psi^{E[t]}(x), n+1+t) \in [n+1+E[t], E[t]+n+2]$. Whereas if $a = (x, n+1) \in X \times [n, n+1]$ then $\tilde{\sigma}_t(a) = (\psi^{E[t]+1}(x), n+1+t) \in X \times [n+1+E[t], E[t]+n+2]$, which is equal to $\tilde{\sigma}_t(b)$ with $b = (\psi(x), n+1) \in X \times [n+1, n+2]$. Therefore $(\tilde{\sigma}_t)_{t \in \mathbf{R}^+}$ descends to the mapping-telescope K_ψ, where it defines a one parameter family $(\sigma_t)_{t \in \mathbf{R}^+}$ of continuous maps of K_ψ. This family depends continuously on the parameter $t \in \mathbf{R}^+$. It satisfies furthermore $\sigma_0 = \mathrm{Id}_{K_\psi}$ and $\sigma_{t+t'} = \sigma_t \circ \sigma_{t'}$. Such a family is called a *semi-flow* on K_ψ.