8. Approximation of straight quasi geodesics in fine position

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

8. Approximation of straight quasi geodesics in fine position

Proposition 8.1. *Let h be a horizontal geodesic. Let g be a straight (J, J')-quasi geodesic, between the orbits of the endpoints of h. There exists a constant $C_{8.1}(|h|_r, J, J')$ such that, if g is in fine position with respect to h, then g is $C_{8.1}(|h|_r, J, J')$-close to the orbit-segments between its endpoints and those of h. Moreover $C_{8.1}(L, J, J') \leq C_{8.1}(M, J, J')$ if $0 \leq L \leq M$, and $C_{8.1}(L, J, J') > C_{8.1}(L', J, J')$ if $L > L' \geq M$.*

Proof. We consider any maximal (in the sense of inclusion) $+$-hole b in g, with $\min_{x \in b} f(x) \geq f(h) + C_{6.7}(J, J')t_0$. By Lemma 6.7, the horizontal geodesic I between its endpoints is dilated in the past after $C_{6.7}(J, J')t_0$ if $|I|_{f(I)} \geq C_{6.7}(J, J')$. Since g and h are in fine position, this implies that $|I|_{f(I)} \leq \max(|h|_r, C_{6.7}(J, J'))$. If $f(h) \leq f(I) \leq f(h) + C_{6.7}(J, J')t_0$, the bounded-dilatation property gives $|I|_{f(I)} \leq \lambda_+^{C_{6.7}(J,J')t_0} |h|_r$.

With the same notation, assume now that b is a maximal $-$-hole with $f(I) \leq f(h) - C_{6.7}(J, J')t_0$. The pulled-tight image of I in the stratum of h is not necessarily contained in h. However, if it is not, then we can write $I = I_1 I_2 I_3$ such that I_1 and I_3 are contained in cancellations, and the pulled-tight image of I_2 in the stratum of h is contained in h. This follows from the fact that h and g are in fine position. If $|I|_{f(I)} \geq C_{6.7}(J, J')$ then, by Lemma 6.7, I is dilated in the future after $C_{6.7}(J, J')t_0$. On the other hand, $|[I_2]_{f(h)}|_{f(h)} \leq |h|_r$, and either $|I_i|_{f(I)} \leq C_{5.3}((C_{6.7}(J, J') + 1)t_0)$ or $|[I_i]_{f(I)+C_{6.7}(J,J')t_0}|_{f(I)+C_{6.7}(J,J')t_0} \leq |I_i|_{f(I)}$ for $i = 1$ or $i = 3$. Indeed $|[I_i]_{f(I)+C_{6.7}(J,J')t_0}|_{f(I)+C_{6.7}(J,J')t_0} > |I_i|_{f(I)} > C_{5.3}((C_{6.7}(J, J') + 1)t_0)$ contradicts Lemma 5.3 since the left inequality implies that $[I_i]_{f(I)+C_{6.7}(J,J')t_0}$ is dilated in the future after t_0, thus I_i would be dilated in the future after $(C_{6.7}(J, J') + 1)t_0$. By Lemma 5.4 we get: If $|I|_{f(I)} \geq C_{6.7}(J, J')$, then

$$|I|_{f(I)} \leq C_{5.4}(C_{6.7}(J, J'), 3, \max(|h|_r, C_{5.3}((C_{6.7}(J, J') + 1)t_0))).$$

It remains to consider the case where $f(h) \geq f(I) \geq f(h) - C_{6.7}(J, J')t_0$. The bounded-cancellation property gives an upper bound for $|I|_{f(I)}$.

We have thus proved that, for any maximal $+$-hole b in g which lies above h, or any maximal $-$-hole b in g which lies below h, the horizontal distance between the endpoints of b is bounded above by some constant $A(|h|_r, J, J')$. Lemmas 7.3 and 7.1 then provide a constant

$$B(|h|_r, J, J') = C_{7.1}(C_{7.3}((A(|h|_r, J, J'), J, J'), C_{7.3}((A(|h|_r, J, J'), J, J'), J, J')$$

such that after replacing maximal $-$-holes in g by the horizontal geodesics between their endpoints, we get a straight $(B(|h|_r, J, J'), B(|h|_r, J, J'))$-quasi geodesic, with the same endpoints, in fine position with respect to h, which is $C_{7.3}(A(|h|_r, J, J'), J, J')$-close to g and which is a stair or the concatenation of two stairs. Lemma 6.4, together with Lemma 5.4 applied as above, then provide $C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J'))$ and

$$D(|h|_r, J, J') = C_{5.4}(1, 3, C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J'))$$

such that this, or these, stair(s) are $D(|h|_r, J, J')$-close to the orbit-segments between h and their endpoints. We conclude that g is $C_{7.3}(A(|h|_r, J, J'), J, J')+ D(|h|_r, J, J')$-close to these orbit-segments. The last point of the proposition is obvious. $\square$

9. Putting paths in fine position

PROPOSITION 9.1. *Let h be a horizontal geodesic. Let g be a straight (J, J')-quasi geodesic, which joins the future or past orbits of the endpoints of h. There exist a constant $C_{9.1}(J, J')$ and a $(C_{9.1}(J, J'), C_{9.1}(J, J'))$-quasi geodesic $\mathcal{G}$ which is $C_{9.1}(J, J')$-close to g, which has the same endpoints as g, and which is in fine position with respect to h.*

Proof. We consider a maximal subpath g' of g whose endpoints lie in the future or past orbits of some points in h, and such that no other point of g' satisfies this property. Consider any maximal $-$-hole b in g', and let I denote the horizontal geodesic between the endpoints of b.

CASE 1. Either I is contained in a cancellation or I is the concatenation of two horizontal geodesics, each contained in a cancellation.

Lemma 6.7 gives $C_{6.7}(J, J')$ such that, if $|I|_{f(I)} \geq C_{6.7}(J, J')$ then I is dilated in the future after $C_{6.7}(J, J')t_0$. Lemma 5.3 gives $C_{5.3}(C_{6.7}(J, J'))$ such that the horizontal length of any horizontal geodesic contained in a cancellation and dilated in the future after $C_{6.7}(J, J')t_0$ is at most $C_{5.3}(C_{6.7}(J, J'))$. By Lemma 5.4 we get an upper bound $C_{5.4}(C_{6.7}(J, J'), 2, C_{5.3}(C_{6.7}(J, J')))$ on the horizontal length of I.

CASE 2. There exists another horizontal geodesic in another connected component of the same stratum whose pulled-tight projection agrees with that of I after some finite time.

We consider the maximal geodesic preimage I' of I under $\sigma_{C_{6.7}(J,J')t_0}$ which connects two points of b. It admits a decomposition into subpaths I'_α