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## 8. APPROXIMATION OF STRAIGHT QUASI GEODESICS IN FINE POSITION

PROPOSITION 8.1. *Let  $h$  be a horizontal geodesic. Let  $g$  be a straight  $(J, J')$ -quasi geodesic, between the orbits of the endpoints of  $h$ . There exists a constant  $C_{8.1}(|h|_r, J, J')$  such that, if  $g$  is in fine position with respect to  $h$ , then  $g$  is  $C_{8.1}(|h|_r, J, J')$ -close to the orbit-segments between its endpoints and those of  $h$ . Moreover  $C_{8.1}(L, J, J') \leq C_{8.1}(M, J, J')$  if  $0 \leq L \leq M$ , and  $C_{8.1}(L, J, J') > C_{8.1}(L', J, J')$  if  $L > L' \geq M$ .*

*Proof.* We consider any maximal (in the sense of inclusion)  $+$ -hole  $b$  in  $g$ , with  $\min_{x \in b} f(x) \geq f(h) + C_{6.7}(J, J')t_0$ . By Lemma 6.7, the horizontal geodesic  $I$  between its endpoints is dilated in the past after  $C_{6.7}(J, J')t_0$  if  $|I|_{f(I)} \geq C_{6.7}(J, J')$ . Since  $g$  and  $h$  are in fine position, this implies that  $|I|_{f(I)} \leq \max(|h|_r, C_{6.7}(J, J'))$ . If  $f(h) \leq f(I) \leq f(h) + C_{6.7}(J, J')t_0$ , the bounded-dilatation property gives  $|I|_{f(I)} \leq \lambda_+^{C_{6.7}(J, J')t_0} |h|_r$ .

With the same notation, assume now that  $b$  is a maximal  $-$ -hole with  $f(I) \leq f(h) - C_{6.7}(J, J')t_0$ . The pulled-tight image of  $I$  in the stratum of  $h$  is not necessarily contained in  $h$ . However, if it is not, then we can write  $I = I_1 I_2 I_3$  such that  $I_1$  and  $I_3$  are contained in cancellations, and the pulled-tight image of  $I_2$  in the stratum of  $h$  is contained in  $h$ . This follows from the fact that  $h$  and  $g$  are in fine position. If  $|I|_{f(I)} \geq C_{6.7}(J, J')$  then, by Lemma 6.7,  $I$  is dilated in the future after  $C_{6.7}(J, J')t_0$ . On the other hand,  $|[I_2]_{f(h)}|_{f(h)} \leq |h|_r$ , and either  $|I_i|_{f(I)} \leq C_{5.3}((C_{6.7}(J, J') + 1)t_0)$  or  $|[I_i]_{f(I) + C_{6.7}(J, J')t_0}]_{f(I) + C_{6.7}(J, J')t_0} \leq |I_i|_{f(I)}$  for  $i = 1$  or  $i = 3$ . Indeed  $|[I_i]_{f(I) + C_{6.7}(J, J')t_0}]_{f(I) + C_{6.7}(J, J')t_0} > |I_i|_{f(I)} > C_{5.3}((C_{6.7}(J, J') + 1)t_0)$  contradicts Lemma 5.3 since the left inequality implies that  $[I_i]_{f(I) + C_{6.7}(J, J')t_0}$  is dilated in the future after  $t_0$ , thus  $I_i$  would be dilated in the future after  $(C_{6.7}(J, J') + 1)t_0$ . By Lemma 5.4 we get: If  $|I|_{f(I)} \geq C_{6.7}(J, J')$ , then

$$|I|_{f(I)} \leq C_{5.4}(C_{6.7}(J, J'), 3, \max(|h|_r, C_{5.3}((C_{6.7}(J, J') + 1)t_0))).$$

It remains to consider the case where  $f(h) \geq f(I) \geq f(h) - C_{6.7}(J, J')t_0$ . The bounded-cancellation property gives an upper bound for  $|I|_{f(I)}$ .

We have thus proved that, for any maximal  $+$ -hole  $b$  in  $g$  which lies above  $h$ , or any maximal  $-$ -hole  $b$  in  $g$  which lies below  $h$ , the horizontal distance between the endpoints of  $b$  is bounded above by some constant  $A(|h|_r, J, J')$ . Lemmas 7.3 and 7.1 then provide a constant

$$B(|h|_r, J, J') = C_{7.1}(C_{7.3}((A(|h|_r, J, J'), J, J'), C_{7.3}((A(|h|_r, J, J'), J, J'), J, J'))$$

such that after replacing maximal  $-$ -holes in  $g$  by the horizontal geodesics between their endpoints, we get a straight  $(B(|h|_r, J, J'), B(|h|_r, J, J'))$ -quasi

geodesic, with the same endpoints, in fine position with respect to  $h$ , which is  $C_{7.3}(A(|h|_r, J, J'), J, J')$ -close to  $g$  and which is a stair or the concatenation of two stairs. Lemma 6.4, together with Lemma 5.4 applied as above, then provide  $C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J'))$  and

$$D(|h|_r, J, J') = C_{5.4}(1, 3, C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J')))$$

such that this, or these, stair(s) are  $D(|h|_r, J, J')$ -close to the orbit-segments between  $h$  and their endpoints. We conclude that  $g$  is  $C_{7.3}(A(|h|_r, J, J'), J, J') + D(|h|_r, J, J')$ -close to these orbit-segments. The last point of the proposition is obvious.  $\square$

### 9. PUTTING PATHS IN FINE POSITION

**PROPOSITION 9.1.** *Let  $h$  be a horizontal geodesic. Let  $g$  be a straight  $(J, J')$ -quasi geodesic, which joins the future or past orbits of the endpoints of  $h$ . There exist a constant  $C_{9.1}(J, J')$  and a  $(C_{9.1}(J, J'), C_{9.1}(J, J'))$ -quasi geodesic  $\mathcal{G}$  which is  $C_{9.1}(J, J')$ -close to  $g$ , which has the same endpoints as  $g$ , and which is in fine position with respect to  $h$ .*

*Proof.* We consider a maximal subpath  $g'$  of  $g$  whose endpoints lie in the future or past orbits of some points in  $h$ , and such that no other point of  $g'$  satisfies this property. Consider any maximal  $-$ -hole  $b$  in  $g'$ , and let  $I$  denote the horizontal geodesic between the endpoints of  $b$ .

**CASE 1.** Either  $I$  is contained in a cancellation or  $I$  is the concatenation of two horizontal geodesics, each contained in a cancellation.

Lemma 6.7 gives  $C_{6.7}(J, J')$  such that, if  $|I|_{f(I)} \geq C_{6.7}(J, J')$  then  $I$  is dilated in the future after  $C_{6.7}(J, J')t_0$ . Lemma 5.3 gives  $C_{5.3}(C_{6.7}(J, J'))$  such that the horizontal length of any horizontal geodesic contained in a cancellation and dilated in the future after  $C_{6.7}(J, J')t_0$  is at most  $C_{5.3}(C_{6.7}(J, J'))$ . By Lemma 5.4 we get an upper bound  $C_{5.4}(C_{6.7}(J, J'), 2, C_{5.3}(C_{6.7}(J, J')))$  on the horizontal length of  $I$ .

**CASE 2.** There exists another horizontal geodesic in another connected component of the same stratum whose pulled-tight projection agrees with that of  $I$  after some finite time.

We consider the maximal geodesic preimage  $I'$  of  $I$  under  $\sigma_{C_{6.7}(J, J')t_0}$  which connects two points of  $b$ . It admits a decomposition into subpaths  $I'_\alpha$