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geodesic, with the same endpoints, in fine position with respect to  $h$ , which is  $C_{7.3}(A(|h|_r, J, J'), J, J')$ -close to  $g$  and which is a stair or the concatenation of two stairs. Lemma 6.4, together with Lemma 5.4 applied as above, then provide  $C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J'))$  and

$$D(|h|_r, J, J') = C_{5.4}(1, 3, C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J')))$$

such that this, or these, stair(s) are  $D(|h|_r, J, J')$ -close to the orbit-segments between  $h$  and their endpoints. We conclude that  $g$  is  $C_{7.3}(A(|h|_r, J, J'), J, J') + D(|h|_r, J, J')$ -close to these orbit-segments. The last point of the proposition is obvious.  $\square$

### 9. PUTTING PATHS IN FINE POSITION

**PROPOSITION 9.1.** *Let  $h$  be a horizontal geodesic. Let  $g$  be a straight  $(J, J')$ -quasi geodesic, which joins the future or past orbits of the endpoints of  $h$ . There exist a constant  $C_{9.1}(J, J')$  and a  $(C_{9.1}(J, J'), C_{9.1}(J, J'))$ -quasi geodesic  $\mathcal{G}$  which is  $C_{9.1}(J, J')$ -close to  $g$ , which has the same endpoints as  $g$ , and which is in fine position with respect to  $h$ .*

*Proof.* We consider a maximal subpath  $g'$  of  $g$  whose endpoints lie in the future or past orbits of some points in  $h$ , and such that no other point of  $g'$  satisfies this property. Consider any maximal  $-$ -hole  $b$  in  $g'$ , and let  $I$  denote the horizontal geodesic between the endpoints of  $b$ .

**CASE 1.** Either  $I$  is contained in a cancellation or  $I$  is the concatenation of two horizontal geodesics, each contained in a cancellation.

Lemma 6.7 gives  $C_{6.7}(J, J')$  such that, if  $|I|_{f(I)} \geq C_{6.7}(J, J')$  then  $I$  is dilated in the future after  $C_{6.7}(J, J')t_0$ . Lemma 5.3 gives  $C_{5.3}(C_{6.7}(J, J'))$  such that the horizontal length of any horizontal geodesic contained in a cancellation and dilated in the future after  $C_{6.7}(J, J')t_0$  is at most  $C_{5.3}(C_{6.7}(J, J'))$ . By Lemma 5.4 we get an upper bound  $C_{5.4}(C_{6.7}(J, J'), 2, C_{5.3}(C_{6.7}(J, J')))$  on the horizontal length of  $I$ .

**CASE 2.** There exists another horizontal geodesic in another connected component of the same stratum whose pulled-tight projection agrees with that of  $I$  after some finite time.

We consider the maximal geodesic preimage  $I'$  of  $I$  under  $\sigma_{C_{6.7}(J, J')t_0}$  which connects two points of  $b$ . It admits a decomposition into subpaths  $I'_\alpha$

connecting points in  $b$  such that the subpath of  $b$  between the endpoints of each  $I'_\alpha$  is a  $-$ -hole. The strong hyperbolicity of the semi-flow implies, by Lemma 6.7, that the horizontal length of each  $I'_\alpha$  is bounded above by  $C_{6.7}(J, J')$ . Since  $g$  is a  $(J, J')$ -quasi geodesic, we get  $\max_{x \in b}(f(I) - f(x)) \leq JC_{6.7}(J, J') + J' + C_{6.7}(J, J')$ .

CASE 3. Some subpath of  $I$  connects the future or past orbits of points in  $h$ .

The only possibility is that  $I$  be a pulled-tight image of  $h$ , i.e.  $g' = b$ . Consider a geodesic preimage  $I'$  of  $I$  under  $\sigma_{C_{6.7}(J, J')t_0}$  between two points in  $b$ . Then proceed as in Case 2, the only difference being that for each subpath  $I_\alpha$ , either there exists a horizontal geodesic in another connected component of the same stratum, whose pulled-tight projection agrees with that of  $I_\alpha$  after some finite time (this is exactly Case 2), or  $I_\alpha$  is contained in a cancellation or in the union of two cancellations, and the arguments are exactly those of Case 1. The bounded-dilatation property then gives an upper bound on the horizontal length of  $I$ .

We denote by  $A(J, J')$  the largest of the constants found in Cases 1, 2 and 3. We denote by  $A'(J, J')$  the largest of the constants  $A(J, J')$ ,  $C_{7.3}(A(J, J'), J, J')$  and  $C_{7.2}(A(J, J'), J, J')$ . Lemmas 7.2, 7.3 and 7.1 then give  $B(J, J') = C_{7.1}(A'(J, J'), A'(J, J'), J, J')$ , such that replacing the maximal  $-$ -holes in  $g'$  by the horizontal geodesic between their endpoints yields a straight  $(B(J, J'), B(J, J'))$ -quasi geodesic stair  $S$ , with the same endpoints, which is  $A'(J, J')$ -close to  $g'$ . Let  $I'$  be a horizontal geodesic between  $S$  and a future or past orbit of some point in  $h$ , which is minimal in the sense of inclusion, i.e. does not contain any subpath connecting  $S$  to a future or past orbit of a point in  $h$ . This horizontal geodesic  $I'$  is a pulled-tight image of a subpath of  $S$  in the stratum considered. It is either contained in a cancellation, or is the union of two horizontal geodesics contained in a cancellation. Lemma 6.4 gives  $C_{6.4}(B(J, J'), B(J, J'))$  such that, if  $|I'|_{f(I')} \geq C_{6.4}(B(J, J'), B(J, J'))$  then  $I'$  is dilated in the futur after  $t_0$ . From Lemmas 5.3 and 5.4 we get  $|I'|_{f(I')} \leq C_{5.4}(1, 2, C_{5.3}(1))$ . Therefore  $S$  is at horizontal distance at most  $D(J, J') = \max(C_{6.4}(B(J, J'), B(J, J')), C_{5.4}(1, 2, C_{5.3}(1)))$  from a straight stair  $S(g')$ , with the same endpoints and in fine position with respect to  $h$ . Lemmas 7.4 and 7.1 then give  $E(J, J') = C_{7.1}(C_{7.4}(D(J, J'), B(J, J'), B(J, J')), C_{7.4}(D(J, J'), B(J, J'), B(J, J')), J, J')$  such that replacing the maximal subpaths  $g'$  as above by the given stair  $S(g')$  gives a straight  $(E(J, J'), E(J, J'))$ -quasi geodesic, with the same endpoints as  $g$ , in fine position with respect to  $h$ , and which is  $D(J, J')$ -close to  $g$ .  $\square$