Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HYPERBOLICITY OF MAPPING-TORUS GROUPS AND SPACES
Autor: Gautero, François
Kapitel: 10. Straight quasi geodesic bigons are thin
DOI: https://doi.org/10.5169/seals-66690

10. STRAIGHT QUASI GEODESIC BIGONS ARE THIN

PROPOSITION 10.1. *There exists a constant $Bi(J,J')$ such that any straight (J,J')-quasi geodesic bigon is $Bi(J,J')$-thin.*

Proof. We denote by g, g' the two sides of a (J,J')-quasi geodesic bigon. We assume for a while that some horizontal geodesic connects the past orbits of the endpoints of the bigon. We choose such a horizontal geodesic h satisfying $f(h) \leq \min_{x \in g \cup g'} f(x) - C_{9.1}(J,J')$. Proposition 9.1 gives a $(C_{9.1}(J,J'), C_{9.1}(J,J'))$-quasi geodesic bigon, with the same vertices, which is $C_{9.1}(J,J')$-close to $g \cup g'$. We denote the sides of this bigon by $\mathcal{G}$ and $\mathcal{G}'$.

Let us call a *diagonal* a horizontal geodesic which minimizes the horizontal distance between the future and past orbits of its endpoints. From the hyperbolicity of the semi-flow, any diagonal with horizontal length at least M is dilated both in the future and in the past after $2t_0$.

We choose a real number $L_0 \geq C_{5.4}(2,3,\lambda_+^{2t_0} C_{5.3}(2)) \geq M$ (the meaning of the constant $C_{5.4}(2,3,\lambda_+^{2t_0} C_{5.3}(2))$ will become clear later). Let $P \in \mathcal{G}$. We assume that there exist two points $P_1, P_2 \in h$, whose future orbits intersect $\mathcal{G}$, such that P is at telescopic distance $L_1 > C_{8.1}(L_0, C_{9.1}(J,J'), C_{9.1}(J,J'))$ from $O^+(P_i) \cup O^-(P_i)$, $i = 1, 2$.

We consider a diagonal D between $O^+(P_1) \cup O^-(P_1)$ and $O^+(P_2) \cup O^-(P_2)$. This diagonal is in fine position with respect to h. Since $\mathcal{G}$ is in fine position with respect to h, and D connects the future or past orbits of points in h, and the future or past orbits of points in $\mathcal{G}$, then $\mathcal{G}$ is in fine position with respect to D. Since the point P is at telescopic distance $L_1 > C_{8.1}(L_0, C_{9.1}(J,J'), C_{9.1}(J,J'))$ from $O^+(P_1) \cup O^-(P_1)$ and from $O^+(P_2) \cup O^-(P_2)$, Proposition 8.1 implies that $|D|_{f(D)} > L_0$.

Since $\mathcal{G}$ is in fine position with respect to D, and connects the union of the future and past orbits of the endpoints of D, some horizontal geodesics connect $P \in \mathcal{G}$ to $O^+(P_1)$ and to $O^+(P_2)$. Either these horizontal geodesics are contained in the pulled-tight image of D, or some pulled-tight image of their concatenation contains D. Because of the bounded-cancellation and bounded-dilatation properties, the telescopic distance between a point and an orbit tends to infinity with the horizontal distance between this point and that orbit. Since the telescopic distance between P and $O^+(P_1) \cup O^-(P_1)$, and between P and $O^+(P_2) \cup O^-(P_2)$ is L_1, this simple observation gives an upper bound X, depending on L_1, for the horizontal length of each of these horizontal geodesics. Therefore some horizontal geodesic connecting $O^+(P_1) \cup O^-(P_1)$ to $O^+(P_2) \cup O^-(P_2)$ has horizontal length at most equal to some constant $2X$ (depending on L_1). In particular, $|D|_{f(D)} \leq 2X$.

We observed that a diagonal D with $|D|_{f(D)} \geq M$ is dilated both in the future and in the past after $2t_0$. Here $|D|_{f(D)} > L_0 \geq M$. Since the concatenation of the above two horizontal geodesics, which lie in the future or in the past of D, has horizontal length at most $2X$, a straightforward computation gives $Y > 0$, still depending on L_1, such that $|f(P) - f(D)| \leq Y$. Lemma 5.6 then implies that P is at telescopic distance smaller than $C_{5.6}(Y)$ from some point in D.

Since $\mathcal{G}'$ and D are in fine position, if no point of $\mathcal{G}'$ lies in the future or past orbit of an endpoint of D, this endpoint belongs to a cancellation. Thus we can write $D = D_1D_2D_3$, where

- D_1 (resp. D_3) is non trivial if and only if no point of $\mathcal{G}'$ lies in the future or past orbit of the initial (resp. terminal) point of D.
- D_1 and D_3, if non trivial, are contained in cancellations.
- $\mathcal{G}'$ connects the future or past orbits of the endpoints of D_2.

Let us assume that D_1 and D_3 are both trivial. Then, since $2X \geq |D|_{f(D)} \geq L_0$, Proposition 8.1 tells us that some subpath of $\mathcal{G}'$ is $C_{8.1}(2X, C_{9.1}(J,J'), C_{9.1}(J,J'))$-close to the orbit-segments which connect its endpoints to the endpoints of D. We observed that D is dilated both in the future and in the past after $2t_0$. We proved that $2X \geq |D|_{f(D)} \geq L_0$. An easy computation gives a time t_* after which the pulled-tight images and the geodesic preimages of D have horizontal length at least $3C_{8.1}(2X, C_{9.1}(J,J'), C_{9.1}(J,J'))$. Thus some point Q of the above subpath of $\mathcal{G}'$ satisfies $|f(Q) - f(D)| \leq t_*$. Lemma 5.6 gives $C_{5.6}(t_*)$ such that Q is $C_{5.6}(t_*)$-close to D. Therefore $P \in \mathcal{G}$ and $Q \in \mathcal{G}'$ are $C_{5.6}(t_*) + C_{5.6}(Y) + X$-close.

Consider now $D = D_1 D_2 D_3$ with D_1 or D_3 non trivial. Since $|D|_{f(D)} \geq C_{5.4}(2, 3, \lambda_+^{2t_0} C_{5.3}(2))$, and D is dilated in the future after $2t_0$, Lemmas 5.3 and 5.4, together with the bounded-dilatation property, give $|D_2|_{f(D)} \geq \lambda_+^{-2t_0}\lambda_+^{2t_0} C_{5.3}(2) \geq M$. Also obviously $|D_2|_{f(D)} \leq 2X$. As in the case where D_1 and D_3 are trivial, on replacing D by D_2 in the above arguments, Proposition 8.1 and Lemma 5.6 eventually give a constant $C_{5.6}(t_\circ)$ such that some point $Q \in \mathcal{G}'$ is $C_{5.6}(t_\circ)$-close to D_2. Thus $P \in \mathcal{G}$ and $Q \in \mathcal{G}'$ are $C_{5.6}(t_\circ) + C_{5.6}(Y) + X$-close.

Consider now the case in which the points P_1, P_2 do not exist. Then P is L_1-close to some point P' in the orbit of an endpoint, say a, of the bigon. By arguing as above (putting paths in fine position and applying Proposition 8.1), we find a horizontal geodesic h', with one endpoint in the future or past orbit of a, such that both paths $\mathcal{G}$ and $\mathcal{G}'$ have one point A-close to h', for some constant A. Since $\mathcal{G}$ and $\mathcal{G}'$ both end or begin at the point a, this implies that $\mathcal{G}'$ admits a point B-close to each point of the orbit-segment between a and h'. In particular there exists $Q \in \mathcal{G}'$ which is $B + L_1$-close to $P \in \mathcal{G}$.

It remains to consider the case where no horizontal geodesic connects the past orbits of the endpoints of the considered (J, J')-quasi geodesic bigon. Then, in the future orbit of the initial endpoint there exists a point z whose past orbit can be connected to the past orbit of the terminal endpoint, and this property is not satisfied by the point w with $f(z) - f(w) = t_0$, which is either in the future or past orbit of the initial endpoint. The strong hyperbolicity of the semi-flow and Proposition 8.1 then give a constant $C_{8.1}(M, J, J')$ such that initial subpaths of both sides of the bigon are $C_{8.1}(M, J, J') + t_0$-close to the orbit-segment connecting the initial endpoint of the bigon to z. From what precedes, any (R, R')-quasi geodesic bigon between z and the terminal endpoint of the considered bigon is $X(R, R')$-thin, for some constant $X(R, R')$. This easily implies that the given bigon is $2(C_{8.1}(M, J, J') + t_0) + X(R, R' + C_{8.1}(M, J, J') + t_0)$-thin. □

11. Geodesic triangles are thin

The following lemma was suggested to the author by I. Kapovich, and allows us to simplify the conclusion. Let us recall that, in the context of quasi geodesic metric spaces, an (r', s')-chain bigon is a bigon whose sides are (r', s')-chains. Still with this terminology, an (r, s)-chain triangle is a triangle whose sides are (r, s)-chains.

LEMMA 11.1. *Let X be an (r, s)-quasi geodesic metric space. If (r', s')-chain bigons are $\delta(r', s')$-thin, $r' \geq r$, $s' \geq s$, then X is $2\delta(r, 3s)$-hyperbolic.*

Proof. We consider an (r, s)-chain triangle with vertices a, b, c and sides $[ab]$, $[ac]$ and $[bc]$. We consider a point x in the (r, s)-chain $[ab]$ which is closest to c. We claim that $[cx] \cup [xb]$ is an $(r, 3s)$-chain, where $[cx]$ and $[xb]$ denote (r, s)-chains from c to x and from x to b. Indeed, for any points u, v in $[xb]$ or $[cx]$, one obviously has $rd_X(u, v) \geq |[uv]|_X$. Let us