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Proposition 10.1 provides a  $\kappa(r, s) = Bi(A(r, s), A(r, s))$  such that this bigon is  $\kappa(r, s)$ -thin. Thus the given  $(r, s)$ -chain bigon is  $\delta(r, s)$ -thin, with  $\delta(r, s) = \kappa(r, s) + 2A(r, s)$ . By Lemma 11.1, the given forest-stack, which is a  $(1, 2)$ -quasi geodesic metric space, is  $2\delta(1, 6)$ -hyperbolic.  $\square$

## 12. BACK TO MAPPING-TELESCOPES

In this section we elucidate the relationships between forest-stacks and mapping-telescopes.

### 12.1 STATEMENT OF THE THEOREM

An **R-tree** (see [9], [2] among many others) is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an **R-tree** is a topological tree. An **R-forest** is a union of disjoint **R-trees**.

LEMMA 12.1. *Let  $(\Gamma, d_\Gamma)$  be an **R-forest** and let  $\psi: \Gamma \rightarrow \Gamma$  be a forest-map of  $\Gamma$ . Let  $(K_\psi, f, \sigma_t)$  be the mapping-telescope of  $(\psi, \Gamma)$  equipped with a structure of forest-stack as defined in Section 2. Then there is a horizontal metric  $\mathcal{H} = (m_r)_{r \in \mathbf{R}}$  on  $K_\psi$  such that*

1. *The **R-forests**  $(f^{-1}(r), m_r)$  and  $(f^{-1}(r+1), m_{r+1})$  are isometric. Each stratum  $(f^{-1}(n), m_n)$ ,  $n \in \mathbf{Z}$ , is isometric to  $(\Gamma, d_\Gamma)$ .*
2. *For any real  $r$  and any horizontal geodesic  $g \in f^{-1}(r)$ , the map*

$$l_{r,g}: \begin{cases} +1 - r] \rightarrow \mathbf{R}^+ \\ t \mapsto |\sigma_t(g)|_{r+t} \end{cases}.$$

*is monotone.*

*Such a horizontal metric is called a horizontal  $d_\Gamma$ -metric. The telescopic metric associated to a horizontal  $d_\Gamma$ -metric is called a mapping-telescope  $d_\Gamma$ -metric.*

*Proof.* We make each  $\Gamma \times \{n\}$ ,  $n \in \mathbf{Z}$ , an **R-forest** isometric to  $\Gamma$ . We consider a cover of  $\Gamma$  by geodesics of length 1 which intersect only at their endpoints. Each  $\Gamma \times \{n\}$  inherits the same cover. There is a disc  $D_{e,n}$  in  $K_\psi$  for each such horizontal geodesic  $e$  in  $\Gamma \times \{n\}$ . This disc is bounded by  $e$ ,  $\psi(e)$  and the orbit-segments between the endpoints of  $e$  and those of  $\psi(e)$ .

We foliate this disc by segments with endpoints in, and transverse to, the orbit-segments in its boundary. Then we assign a length to each such segment so that the collection of lengths varies continuously and monotonically, from the length of  $e$  to that of  $\psi(e)$ . We thus obtain a horizontal metric on the mapping-telescope. Furthermore each stratum  $f^{-1}(n)$ ,  $n \in \mathbf{Z}$ , is isometric to  $(\Gamma, d_\Gamma)$ . And the maps denoted by  $l_{r,g}$  in Lemma 12.1 are monotone by construction. By definition of a mapping-telescope, the discs  $D_{e,n}$  between  $\Gamma \times \{n\}$  and  $\Gamma \times \{n+1\}$  are copies of the discs  $D_{e,n'}$  between  $\Gamma \times \{n'\}$  and  $\Gamma \times \{n'+1\}$ , for any  $n, n'$  in  $\mathbf{Z}$ . This allows us to choose the horizontal metric to satisfy the further condition that  $(f^{-1}(r), m_r)$  be isometric with  $(f^{-1}(r+1), m_{r+1})$  for any real number  $r$ .  $\square$

We now define dynamical properties for  $\mathbf{R}$ -forest maps.

DEFINITION 12.2. Let  $(\Gamma, d_\Gamma)$  be an  $\mathbf{R}$ -forest. A forest-map  $\psi$  of  $(\Gamma, d_\Gamma)$  is *weakly bi-Lipschitz* if there exist  $\mu \geq 1$  and  $K \geq 0$  such that  $\mu d_\Gamma(x, y) \geq d_\Gamma(\psi(x), \psi(y)) \geq \frac{1}{\mu} d_\Gamma(x, y) - K$ .

DEFINITION 12.3. Let  $(\Gamma, d_\Gamma)$  be an  $\mathbf{R}$ -forest. A forest-map  $\psi$  of  $(\Gamma, d_\Gamma)$  is *hyperbolic* if it is weakly bi-Lipschitz and there exist  $\lambda > 1$ ,  $N \geq 1$ ,  $M \geq 0$  such that for any pair of points  $x, y$  in  $\Gamma$  with  $d_\Gamma(x, y) \geq M$ , either  $d_\Gamma(\psi^N(x), \psi^N(y)) \geq \lambda d_\Gamma(x, y)$  or  $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$  for some  $x_N, y_N$  with  $\psi^N(x_N) = x$ ,  $\psi^N(y_N) = y$ .

A hyperbolic forest-map  $\psi$  of  $(\Gamma, d_\Gamma)$  is *strongly hyperbolic* if, for any pair of points  $x, y$  with  $d_\Gamma(x, y) \geq M$  and each connected component containing both a preimage of  $x$  and a preimage of  $y$  under  $\psi^N$ , there is at least one pair of such preimages  $x_N, y_N$  for which  $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$ .

If the forest  $\Gamma$  is a tree then a hyperbolic forest-map is strongly hyperbolic (similarly we saw that a hyperbolic semi-flow on a forest-stack whose strata are connected is strongly hyperbolic).

Our theorem about mapping-telescopes is

THEOREM 12.4. Let  $(\Gamma, d_\Gamma)$  be an  $\mathbf{R}$ -forest. Let  $\psi$  be a strongly hyperbolic forest-map of  $(\Gamma, d_\Gamma)$  whose mapping-telescope  $K_\psi$  is connected. Then  $K_\psi$  is a Gromov-hyperbolic metric space for any mapping-telescope  $d_\Gamma$ -metric.

## 12.2 PROOF OF THEOREM 12.4

LEMMA 12.5. *Let  $(\Gamma, d_\Gamma)$  be an  $\mathbf{R}$ -forest. Let  $\psi$  be a weakly bi-Lipschitz forest-map of  $(\Gamma, d_\Gamma)$ . Let  $(K_\psi, f, \sigma_t)$  be the mapping-telescope of  $(\psi, \Gamma)$ , equipped with a structure of forest-stack as defined in Section 2. Then the semi-flow  $(\sigma_t)_{t \in \mathbf{R}^+}$  is a bounded-cancellation and bounded-dilatation semi-flow with respect to any horizontal  $d_\Gamma$ -metric (see Lemma 12.1).*

*Proof.* The horizontal metric  $\mathcal{H}$  agrees with the metric  $d_\Gamma$  on all the strata  $f^{-1}(n)$ ,  $n \in \mathbf{Z}$  (see Lemma 12.1). Consider any horizontal geodesic  $g$  in the stratum  $f^{-1}(0)$ . If  $\psi$  is weakly bi-Lipschitz with constants  $\mu_0$  and  $K_0$ , then for any integer  $n \geq 0$ , we have  $|[g]_n|_n \geq \frac{1}{\mu_0^n} |g|_0 - K_0 \left( \frac{1}{\mu_0^{n-1}} + \frac{1}{\mu_0^{n-2}} + \dots + 1 \right)$ . Since  $0 < \frac{1}{\mu_0} < 1$ , the sum tends to  $\frac{\mu_0}{\mu_0 - 1}$  as  $n \rightarrow +\infty$ . Setting  $\lambda_- = \frac{1}{\mu_0}$  and  $K = K_0 \frac{\mu_0}{\mu_0 - 1}$ , this proves the inequality of item (1) for horizontal geodesics in  $f^{-1}(n)$ ,  $n \in \mathbf{Z}$ , and an integer time  $t$ . For the case in which  $t$  is any positive real number and  $g \in f^{-1}(r)$ ,  $r$  any real number, just decompose  $\sigma_t = \sigma_{t-E[t]} \circ \sigma_{E[t-(E[r]+1-r)]} \circ \sigma_{E[r]+1-r}$ . The map  $\sigma_t$  is a homeomorphism from  $f^{-1}(r)$  onto  $f^{-1}(r+t)$  for any  $t \in [0, E[r]+1-r]$ . That is, for any real  $r$ ,  $|[g]_{r+t}|_{r+t} = |\sigma_t(g)|_{r+t}$  for  $t \in [0, E[r]+1-r]$ . The monotonicity of the maps  $l_{r,g}$  (see Lemma 12.1, item (2)) implies, for any  $r$  and  $t \in [0, E[r]+1-r]$ , that  $|\sigma_t(g)|_{r+t} \geq \frac{1}{\mu_0} |g|_r$ . The conclusion follows.  $\square$

LEMMA 12.6. *With the assumptions and notation of Lemma 12.5, if the map  $\psi$  is a (strongly) hyperbolic forest-map of  $(\Gamma, d_\Gamma)$  then the semi-flow  $(\sigma_t)_{t \in \mathbf{R}^+}$  is (strongly) hyperbolic with respect to any horizontal  $d_\Gamma$ -metric.*

The proof is similar to that of Lemma 12.5.  $\square$

*Proof of Theorem 12.4.* By Lemmas 12.5 and 12.6, a mapping-telescope admits a structure of forest-stack  $(\tilde{X}, f, \sigma_t, \mathcal{H})$  with horizontal metric  $\mathcal{H}$  such that the semi-flow  $(\sigma_t)_{t \in \mathbf{R}^+}$  is a strongly hyperbolic semi-flow with respect to  $\mathcal{H}$ . Hence Theorem 4.4 implies Theorem 12.4.  $\square$

## 13. ABOUT MAPPING-TORUS GROUPS

We first recall the definition of a *hyperbolic endomorphism* of a group introduced by Gromov [19].