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Autor: Gautero, François

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12.2 Proof of Theorem 12.4

LEMMA 12.5. *Let (Γ, d_Γ) be an $\mathbf{R}$-forest. Let ψ be a weakly bi-Lipschitz forest-map of (Γ, d_Γ). Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ), equipped with a structure of forest-stack as defined in Section 2. Then the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is a bounded-cancellation and bounded-dilatation semi-flow with respect to any horizontal d_Γ-metric (see Lemma 12.1).*

Proof. The horizontal metric $\mathcal{H}$ agrees with the metric d_Γ on all the strata $f^{-1}(n)$, $n \in \mathbf{Z}$ (see Lemma 12.1). Consider any horizontal geodesic g in the stratum $f^{-1}(0)$. If ψ is weakly bi-Lipschitz with constants μ_0 and K_0, then for any integer $n \geq 0$, we have $|[g]_n|_n \geq \frac{1}{\mu_0^n}|g|_0 - K_0(\frac{1}{\mu_0^{n-1}} + \frac{1}{\mu_0^{n-2}} + \ldots + 1)$. Since $0 < \frac{1}{\mu_0} < 1$, the sum tends to $\frac{\mu_0}{\mu_0 - 1}$ as $n \to +\infty$. Setting $\lambda_- = \frac{1}{\mu_0}$ and $K = K_0 \frac{\mu_0}{\mu_0 - 1}$, this proves the inequality of item (1) for horizontal geodesics in $f^{-1}(n)$, $n \in \mathbf{Z}$, and an integer time t. For the case in which t is any positive real number and $g \in f^{-1}(r)$, r any real number, just decompose $\sigma_t = \sigma_{t-E[t]} \circ \sigma_{E[t-(E[r]+1-r)]} \circ \sigma_{E[r]+1-r}$. The map σ_t is a homeomorphism from $f^{-1}(r)$ onto $f^{-1}(r+t)$ for any $t \in [0, E[r]+1-r)$. That is, for any real r, $|[g]_{r+t}|_{r+t} = |\sigma_t(g)|_{r+t}$ for $t \in [0, E[r]+1-r)$. The monotonicity of the maps $l_{r,g}$ (see Lemma 12.1, item (2)) implies, for any r and $t \in [0, E[r]+1-r)$, that $|\sigma_t(g)|_{r+t} \geq \frac{1}{\mu_0}|g|_r$. The conclusion follows. $\square$

LEMMA 12.6. *With the assumptions and notation of Lemma 12.5, if the map ψ is a (strongly) hyperbolic forest-map of (Γ, d_Γ) then the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is (strongly) hyperbolic with respect to any horizontal d_Γ-metric.*

The proof is similar to that of Lemma 12.5. $\square$

Proof of Theorem 12.4. By Lemmas 12.5 and 12.6, a mapping-telescope admits a structure of forest-stack $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ with horizontal metric $\mathcal{H}$ such that the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is a strongly hyperbolic semi-flow with respect to $\mathcal{H}$. Hence Theorem 4.4 implies Theorem 12.4. $\square$

13. About Mapping-Torus Groups

We first recall the definition of a *hyperbolic endomorphism* of a group introduced by Gromov [19].

DEFINITION 13.1 ([19], [3]). An injective endomorphism α of the rank n free group F_n is hyperbolic if there exist $\lambda_\alpha > 1$ and $j_\alpha > 0$ such that for any $w \in F_n$, either $\lambda_\alpha |w| \leq |\alpha^{j_\alpha}(w)|$ or w admits a preimage $\alpha^{-j_\alpha}(w)$ such that $\lambda_\alpha |w| \leq |\alpha^{-j_\alpha}(w)|$, where $|\,.\,|$ denotes the usual word-metric.

We recall that a subgroup H in a group G is *malnormal* if $w^{-1}Hw \cap H = \{1\}$ for any element $w \notin H$ of G. We state our theorem about mapping-torus groups as follows:

THEOREM 13.2. *Let α be an injective hyperbolic endomorphism of the rank n free group F_n. If the image of α is a malnormal subgroup of F_n then the mapping-torus group $G_\alpha = \langle x_1, \ldots, x_n, t \,;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \ldots, n \rangle$ is a hyperbolic group.*

13.1 RELATIONSHIPS WITH MAPPING-TELESCOPES

We consider the rank n free group $F_n = \langle x_1, \ldots, x_n \rangle$. Let α be an injective endomorphism of F_n. Let $G_\alpha = \langle x_1, \ldots, x_n, t \,;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \ldots, n \rangle$ be the mapping-torus group of (α, F_n). We consider the Cayley graph Γ associated to the given system of generators. Let l be a loop in Γ whose associated word in the edges of Γ reads a relation $t^{-1}x_i t \alpha(x_i)^{-1}$. We attach a 2-cell by its boundary circle along any such loop l. The resulting topological space is a 2-complex. This is the Cayley complex of the mapping-torus group G_α for the given presentation.

Let us check that the above Cayley complex is a mapping-telescope of a forest-map. We consider the rose $\mathcal{R}_n$ with n petals. We label each edge by a generator x_i of F_n. We denote by ψ the simplicial map on $\mathcal{R}_n$ such that $\psi(x_i)$ is a locally injective path whose associated word in the edges of $\mathcal{R}_n$ reads $\alpha(x_i)$. Let us denote by T the universal covering of $\mathcal{R}_n$ (T is a tree) and by $\pi \colon T \to \mathcal{R}_n$ the associated covering-map. We denote by $\widehat{\psi} \colon T \to T$ a simplicial lift of ψ to T, that is $\pi \circ \widehat{\psi} = \psi \circ \pi$. We consider the mapping-torus of $(\psi, \mathcal{R}_n)$, i.e. the 2-complex $\mathcal{R}_n \times [0,1]/(x,1) \sim (\psi(x), 0)$. Then the universal covering of this mapping-torus is the mapping-telescope of $\widetilde{\psi} \colon F \to F$, where F and $\widetilde{\psi}$ are defined as follows:

- We denote by I the set of integers from 1 to $\mathrm{Card}(F_n/\,\mathrm{Im}(\alpha))$. The different classes are written $w_i \mathrm{Im}(\alpha)$, $i = 0, 1, \ldots$. We denote by $\gamma \colon I \to \{w_0, w_1, \ldots\}$ the bijection. Then the connected components of F are in bijection with $\mathbf{N}^{\mathrm{Card}(I)}$. Each connected component is the image, by a bijection μ, of a sequence of Card(I) integers. Each connected component $\mu(x_0, x_1, \dots)$ of F is homeomorphic to T via $\beta_{(x_0,x_1,\dots)}\colon \mu(x_0, x_1, \dots) \to T$.

• We define the restriction of $\widetilde{\psi}$ to any connected component $\mu((x_0, x_1, \dots))$ as follows:

If $\mathrm{Card}(I) < +\infty$ then

$$\widetilde{\psi}|_{\mu((x_0,x_1,\dots))}\colon \begin{cases} \mu((x_0, x_1, \dots)) & \to \quad \mu((E[\frac{x_0}{\mathrm{Card}(I)}], x_1, \dots)) \\ x & \to \quad (\gamma(j)\beta^{-1}_{(x_0,x_1,\dots)}\widehat{\psi}\beta_{(x_0,x_1,\dots)})(x) \end{cases}$$

where $j < \mathrm{Card}(I)$ satisfies $E[\frac{x_0}{\mathrm{Card}(I)}] = k\,\mathrm{Card}(I) + j$.

If $\mathrm{Card}(I) = +\infty$ then

$$\widetilde{\psi}|_{\mu((x_0,x_1,\dots))}\colon \begin{cases} \mu((x_0, x_1, \dots)) & \to \quad \mu((x_1, x_2, \dots)) \\ x & \to \quad (\gamma(x_0)\beta^{-1}_{(x_0,x_1,\dots)}\widehat{\psi}\beta_{(x_0,x_1,\dots)})(x)\,. \end{cases}$$

The mapping-torus of $(\psi, \mathcal{R}_n)$ is a 2-complex whose 1-skeleton is the rose with $n+1$ petals in bijection with $\{x_1, \dots, x_n, t\}$. There is one 2-cell for each relation $t^{-1}x_i t\alpha(x_i)^{-1}$. Thus the universal covering described above is the Cayley complex for G_α with the presentation $G_\alpha = \langle x_1, \dots, x_n, t\,;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \dots, n\rangle$. We have thus proved

LEMMA 13.3. *Let α be an injective endomorphism of $F_n = \langle x_1, \dots, x_n\rangle$. Let $G_\alpha = \langle x_1, \dots, x_n, t\,;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \dots, n\rangle$ be the mapping-torus group of α. Let $\mathcal{C}(G_\alpha)$ be the Cayley complex of G_α for the given presentation. Then $\mathcal{C}(G_\alpha)$ is the mapping-telescope of a forest-map.*

REMARK 13.4. If the endomorphism α is an automorphism then the above Cayley complex is the mapping-telescope of a tree-map. The tree is the universal covering of the rose with n petals. If the endomorphism α is not injective then some element $w \in F_n$ satisfies $w = 1$ in G_α; the above construction fails because of the corresponding loops in the Cayley graph.

Let α be an injective free group endomorphism. Let G_α be the mapping-torus group of α. Let $\mathcal{C}(G_\alpha)$ be the Cayley complex of G_α for the usual presentation $G_\alpha = \langle x_1, \dots, x_n, t\,;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \dots, n\rangle$. By Lemma 13.3, $\mathcal{C}(G_\alpha)$ is a mapping-telescope of a forest-map. We now want to see what happens with respect to metrics and dynamics. The Cayley graph of a group is equipped with a metric which makes each edge isometric to the interval $(0, 1)$. More generally, given a graph Γ, we call *standard metric*, and denote by d^s_Γ, such a metric on Γ. We will call *mapping-telescope standard metric* any mapping-telescope d^s_Γ-metric on $\mathcal{C}(G_\alpha)$.

LEMMA 13.5. *The mapping-torus group G_α of an injective free group endomorphism acts cocompactly, properly discontinuously and isometrically on the Cayley complex $\mathcal{C}(G_\alpha)$ equipped with any mapping-telescope standard metric.*

Proof. We consider the usual action by left translations of the group on its Cayley graph. This action is extended in a natural way to a free action on the Cayley complex $\mathcal{C}(G_\alpha)$. Let f denote the map giving the strata for the structure of forest-stack of $\mathcal{C}(G_\alpha)$, see Lemma 13.3. For a mapping-telescope metric, all the strata $f^{-1}(r)$ and $f^{-1}(r+1)$ are isometric. And for a mapping-telescope standard metric all the strata $f^{-1}(n)$, $n \in \mathbf{Z}$, are equipped with the standard metric. This readily implies that the above action is isometric. □

13.2 FREE GROUP ENDOMORPHISMS AND FOREST-MAPS

The main point of Lemma 13.6 below is the so-called 'bounded-cancellation lemma' of [7] for free group automorphisms, and of [10] for the injective free group endomorphisms.

LEMMA 13.6. *Let α be an injective free group endomorphism. Let F and $\widetilde{\psi}$ be the forest and the forest-map on F given by Lemma 13.3. Then $\widetilde{\psi}$ is a weakly bi-Lipschitz forest-map of F equipped with the standard metric d^s_F.*

Proof. If w is any element in $F_n = \langle x_1, \ldots, x_n \rangle$, and $|\,.\,|_{F_n}$ denotes the word-metric on F_n, then $|\alpha(w)|_{F_n} \leq (\max_{i=1,\ldots,n} |\alpha(x_i)|_{F_n})|w|_{F_n}$. By definition of the standard metric, and setting $\mu_0 = \max_{i=1,\ldots,n} |\alpha(x_i)|_{F_n}$, the map $\widetilde{\psi}$ satisfies $d^s_F(\widetilde{\psi}(x), \widetilde{\psi}(y)) \leq \mu_0 d^s_F(x,y)$ for any pair of *vertices* x, y. If x, y are not vertices, then they are joined in their stratum by a horizontal geodesic which is the concatenation of a path between two vertices, with two proper subsets of edges. By construction and simpliciality of $\widetilde{\psi}$, proper subsets of edges are dilated by a bounded factor when applying $\widetilde{\psi}$, so that the conclusion follows for the upper bound.

If w is any element in F_n then

$$\left|\alpha^{-1}(w)\right|_{F_n} \leq (\max_{i=1,\ldots,n} \left|\alpha^{-1}(x_i)\right|_{F_n})|w|_{F_n}.$$

Setting $\mu_1 = \max_{i=1,\ldots,n} \left|\alpha^{-1}(x_i)\right|_{F_n}$ we get $|\alpha(w)|_{F_n} \geq \frac{1}{\mu_1}|w|_{F_n}$. Therefore $d^s_F(\widetilde{\psi}(x), \widetilde{\psi}(y)) \geq \frac{1}{\mu_1} d^s_F(x,y)$ for any pair of *vertices* x, y. The inequality for all points x, y does not follow as easily as for the upper bound, since the map $\widetilde{\psi}$ might identify points, and this could make the distance decrease sharply. However, assume the existence of a constant K_0 such that $\widetilde{\psi}(x) = \widetilde{\psi}(y) \Rightarrow d^s_F(x,y) \leq K_0$. Any geodesic in F is the concatenation of a geodesic between two vertices with two proper subsets of edges of F. Thus the inequality $d^s_F(\widetilde{\psi}(x), \widetilde{\psi}(y)) \geq \frac{1}{\mu_1} d^s_F(x,y) - 2K_0$ follows in a straightforward way from the preceding assertions. Injective free group endomorphisms satisfy the so-called 'bounded-cancellation lemma' (see [10], and [7] for the particular case of automorphisms), i.e. there exists $A_\alpha > 0$ such that $|\alpha(w_1 w_2)|_{F_n} \geq |\alpha(w_1)|_{F_n} + |\alpha(w_2)|_{F_n} - A_\alpha$ for any w_1, w_2 in F_n with $|w_1 w_2|_{F_n} = |w_1|_{F_n} + |w_2|_{F_n}$. This inequality gives a constant $K_0 = A_\alpha + 2$ as required above, i.e. such that, if $\widetilde{\psi}(x) = \widetilde{\psi}(y)$ then $d^s_F(x,y) \leq K_0$. Setting $\mu = \max(\mu_0, \mu_1)$ and $K = 2K_0$, we get Lemma 13.6. □

LEMMA 13.7. *With the assumptions and notation of Lemma 13.6,*

1) *If α is hyperbolic then the forest-map is hyperbolic.*

2) *If α is hyperbolic and its image* $\mathrm{Im}(\alpha)$ *is malnormal, then the forest-map is strongly hyperbolic.*

Proof. (1) is easy to check. Let us prove (2). The notation used is that introduced in Section 13 when defining the forest F and the map $\widetilde{\psi}$. If the map is not strongly hyperbolic, there exists an infinite sequence of pairs of connected components (T_i, T'_i) such that T_i and T'_i are identified under $\widetilde{\psi}$ along a geodesic g_i and the length of g_i tends to $+\infty$ as $i \to +\infty$. Thus there exists an infinite number of elements $(u_i, u'_i) \in F_n - \mathrm{Im}(\alpha) \times F_n - \mathrm{Im}(\alpha)$ such that some geodesic word $a_i w_i b_i$ (resp. $a'_i w_i b'_i$) connects two vertices associated to elements in $u_i \mathrm{Im}(\alpha)$ (resp. in $u'_i \mathrm{Im}(\alpha)$) where the length of the w_i's tends to $+\infty$ as $i \to +\infty$.

Observe that in particular $a_i w_i b_i \in \mathrm{Im}(\alpha)$, $a'_i w_i b'_i \in \mathrm{Im}(\alpha)$, whereas $a_i w_i b'_i \notin \mathrm{Im}(\alpha)$ and $a'_i w_i b_i \notin \mathrm{Im}(\alpha)$ because they carry an element of $u_i \mathrm{Im}(\alpha)$ (resp. $u'_i \mathrm{Im}(\alpha)$) to an element of $u'_i \mathrm{Im}(\alpha)$ (resp. of $u_i \mathrm{Im}(\alpha)$). The lengths of the a_i, b_i, a'_i, b'_i can be assumed to be at most the maximum of the lengths of the images under α of the generators of F_n, which is finite. Since there are only a finite number of pairs of elements of bounded lengths, a same pair a_I, b_I (resp. a'_I, b'_I) appears an infinite number of times when listing the sequence of words $a_i w_i b_i$ (resp. $a'_i w_i b'_i$). The same finiteness argument then gives two words $\omega_1 \subsetneq \omega_2$ with $\omega_2 = \omega\omega_1$ such that $a_I \omega_j b_I \in \mathrm{Im}(\alpha)$, $a'_I \omega_j b'_I \in \mathrm{Im}(\alpha)$, $a_I \omega_j b'_I \notin \mathrm{Im}(\alpha)$ and $a'_I \omega_j b_I \notin \mathrm{Im}(\alpha)$, $j = 1, 2$.

Thus $a_I\omega_1 b_I b_I^{-1}\omega_1^{-1}\omega^{-1}a_I^{-1} \in \mathrm{Im}(\alpha)$, $a'_I\omega_1 b'_I {b'}_I^{-1}\omega_1^{-1}\omega^{-1}{a'}_I^{-1} \in \mathrm{Im}(\alpha)$, $a_I\omega_1 b'_I {b'}_I^{-1}\omega_1^{-1}\omega^{-1}{a'}_I^{-1} \notin \mathrm{Im}(\alpha)$. Now $(a_I\omega^{-1}{a'}_I^{-1})^{-1} a_I\omega^{-1}a_I^{-1}(a_I\omega^{-1}{a'}_I^{-1}) = a'_I\omega^{-1}{a'}_I^{-1} \in \mathrm{Im}(\alpha)$, whereas $a_I\omega^{-1}{a'}_I^{-1} \notin \mathrm{Im}(\alpha)$ and $a_I\omega^{-1}a_I^{-1} \in \mathrm{Im}(\alpha)$. We thus get a contradiction to the malnormality of $\mathrm{Im}(\alpha)$ in F_n. This completes the proof. □

13.3 Proof of Theorem 13.2

From Lemmas 13.6 and 13.7, the Cayley complex $\mathcal{C}(G_\alpha)$ is the mapping-telescope of a strongly hyperbolic forest-map, equipped with the standard metric. A Cayley complex is connected. Thus, from Theorem 12.4, $\mathcal{C}(G_\alpha)$ is a Gromov-hyperbolic metric space for any mapping-telescope standard metric. From Lemma 13.5 the group G_α acts cocompactly, properly discontinuously and isometrically on $\mathcal{C}(G_\alpha)$ equipped with a mapping-telescope standard metric. A classical lemma of geometric group theory (usually attributed to Effremovich, Svàrc, Milnor – see [19] or [17] for instance), applied to quasi geodesic metric spaces, tells us that G_α and $\mathcal{C}(G_\alpha)$ are quasi-isometric so that G_α is a hyperbolic group. □

Remark 13.8. Another way of stating our main theorem about 'forest-stacks', using the language of trees of spaces, goes roughly as follows: "An oriented $\mathbf{R}$-tree of $\mathbf{R}$-trees with the gluing-maps satisfying the conditions of hyperbolicity and strong hyperbolicity with uniform constants is Gromov-hyperbolic." Here 'oriented $\mathbf{R}$-tree' means an $\mathbf{R}$-tree T equipped with an orientation going from the domain to the image of each attaching-map, and a surjective continuous map $f\colon T \to \mathbf{R}$ respecting this orientation. As a corollary of our theorem, and in order to illustrate it, we chose to concentrate on mapping-telescopes. We could as well consider spaces similar to mapping-telescopes but where we allow the attaching-maps not to be the same at each step. Our only requirement is to have uniform constants of quasi-isometry, hyperbolicity and so on. Also, with respect to groups, a corollary could have been stated dealing with HNN-extensions rather than just semi-direct products.

Another result which easily follows from our work could be more or less stated as follows. "Let T be a tree of spaces X_i, $i = 0, 1, \ldots$. Let $\psi\colon T \to T$ be a map of T such that the mapping-telescope of each X_i under ψ is Gromov-hyperbolic. If ψ induces a hyperbolic map on the tree resulting of the collapsing of each X_i to a point, then the mapping-telescope of the tree of spaces T under ψ is Gromov-hyperbolic." We leave the precise statement of such corollaries to the reader. Together with [14] where a new proof of the

Bestvina-Feighn theorem is given for mapping-tori of surface groups, the last one gives, thanks to [26], a new proof of the full version of the Combination Theorem for mapping-tori of hyperbolic groups, namely: "If G is a hyperbolic group and α is a hyperbolic automorphism of G, then $G \rtimes_{\alpha} \mathbf{Z}$ is a hyperbolic group."

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