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DEFINITION 13.1 ([19], [3]). An injective endomorphism  $\alpha$  of the rank  $n$  free group  $F_n$  is hyperbolic if there exist  $\lambda_\alpha > 1$  and  $j_\alpha > 0$  such that for any  $w \in F_n$ , either  $\lambda_\alpha |w| \leq |\alpha^{j_\alpha}(w)|$  or  $w$  admits a preimage  $\alpha^{-j_\alpha}(w)$  such that  $\lambda_\alpha |w| \leq |\alpha^{-j_\alpha}(w)|$ , where  $|\cdot|$  denotes the usual word-metric.

We recall that a subgroup  $H$  in a group  $G$  is *malnormal* if  $w^{-1}Hw \cap H = \{1\}$  for any element  $w \notin H$  of  $G$ . We state our theorem about mapping-torus groups as follows:

THEOREM 13.2. *Let  $\alpha$  be an injective hyperbolic endomorphism of the rank  $n$  free group  $F_n$ . If the image of  $\alpha$  is a malnormal subgroup of  $F_n$  then the mapping-torus group  $G_\alpha = \langle x_1, \dots, x_n, t ; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$  is a hyperbolic group.*

### 13.1 RELATIONSHIPS WITH MAPPING-TELESCOPES

We consider the rank  $n$  free group  $F_n = \langle x_1, \dots, x_n \rangle$ . Let  $\alpha$  be an injective endomorphism of  $F_n$ . Let  $G_\alpha = \langle x_1, \dots, x_n, t ; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$  be the mapping-torus group of  $(\alpha, F_n)$ . We consider the Cayley graph  $\Gamma$  associated to the given system of generators. Let  $l$  be a loop in  $\Gamma$  whose associated word in the edges of  $\Gamma$  reads a relation  $t^{-1}x_it\alpha(x_i)^{-1}$ . We attach a 2-cell by its boundary circle along any such loop  $l$ . The resulting topological space is a 2-complex. This is the Cayley complex of the mapping-torus group  $G_\alpha$  for the given presentation.

Let us check that the above Cayley complex is a mapping-telescope of a forest-map. We consider the rose  $\mathcal{R}_n$  with  $n$  petals. We label each edge by a generator  $x_i$  of  $F_n$ . We denote by  $\psi$  the simplicial map on  $\mathcal{R}_n$  such that  $\psi(x_i)$  is a locally injective path whose associated word in the edges of  $\mathcal{R}_n$  reads  $\alpha(x_i)$ . Let us denote by  $T$  the universal covering of  $\mathcal{R}_n$  ( $T$  is a tree) and by  $\pi: T \rightarrow \mathcal{R}_n$  the associated covering-map. We denote by  $\widehat{\psi}: T \rightarrow T$  a simplicial lift of  $\psi$  to  $T$ , that is  $\pi \circ \widehat{\psi} = \psi \circ \pi$ . We consider the mapping-torus of  $(\psi, \mathcal{R}_n)$ , i.e. the 2-complex  $\mathcal{R}_n \times [0, 1]/(x, 1) \sim (\psi(x), 0)$ . Then the universal covering of this mapping-torus is the mapping-telescope of  $\widetilde{\psi}: F \rightarrow F$ , where  $F$  and  $\widetilde{\psi}$  are defined as follows:

- We denote by  $I$  the set of integers from 1 to  $\text{Card}(F_n/\text{Im}(\alpha))$ . The different classes are written  $w_i \text{Im}(\alpha)$ ,  $i = 0, 1, \dots$ . We denote by  $\gamma: I \rightarrow \{w_0, w_1, \dots\}$  the bijection. Then the connected components of  $F$  are in bijection with  $\mathbf{N}^{\text{Card}(I)}$ . Each connected component is the image, by a

bijection  $\mu$ , of a sequence of  $\text{Card}(I)$  integers. Each connected component  $\mu(x_0, x_1, \dots)$  of  $F$  is homeomorphic to  $T$  via  $\beta_{(x_0, x_1, \dots)}: \mu(x_0, x_1, \dots) \rightarrow T$ .

• We define the restriction of  $\tilde{\psi}$  to any connected component  $\mu((x_0, x_1, \dots))$  as follows:

If  $\text{Card}(I) < +\infty$  then

$$\tilde{\psi}|_{\mu((x_0, x_1, \dots))}: \begin{cases} \mu((x_0, x_1, \dots)) & \rightarrow \mu((E[\frac{x_0}{\text{Card}(I)}], x_1, \dots)) \\ x & \rightarrow (\gamma(j)\beta_{(x_0, x_1, \dots)}^{-1})\hat{\psi}\beta_{(x_0, x_1, \dots)}(x) \end{cases}$$

where  $j < \text{Card}(I)$  satisfies  $E[\frac{x_0}{\text{Card}(I)}] = k\text{Card}(I) + j$ .

If  $\text{Card}(I) = +\infty$  then

$$\tilde{\psi}|_{\mu((x_0, x_1, \dots))}: \begin{cases} \mu((x_0, x_1, \dots)) & \rightarrow \mu((x_1, x_2, \dots)) \\ x & \rightarrow (\gamma(x_0)\beta_{(x_0, x_1, \dots)}^{-1})\hat{\psi}\beta_{(x_0, x_1, \dots)}(x). \end{cases}$$

The mapping-torus of  $(\psi, \mathcal{R}_n)$  is a 2-complex whose 1-skeleton is the rose with  $n + 1$  petals in bijection with  $\{x_1, \dots, x_n, t\}$ . There is one 2-cell for each relation  $t^{-1}x_it\alpha(x_i)^{-1}$ . Thus the universal covering described above is the Cayley complex for  $G_\alpha$  with the presentation  $G_\alpha = \langle x_1, \dots, x_n, t; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$ . We have thus proved

LEMMA 13.3. *Let  $\alpha$  be an injective endomorphism of  $F_n = \langle x_1, \dots, x_n \rangle$ . Let  $G_\alpha = \langle x_1, \dots, x_n, t; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$  be the mapping-torus group of  $\alpha$ . Let  $\mathcal{C}(G_\alpha)$  be the Cayley complex of  $G_\alpha$  for the given presentation. Then  $\mathcal{C}(G_\alpha)$  is the mapping-telescope of a forest-map.*

REMARK 13.4. If the endomorphism  $\alpha$  is an automorphism then the above Cayley complex is the mapping-telescope of a tree-map. The tree is the universal covering of the rose with  $n$  petals. If the endomorphism  $\alpha$  is not injective then some element  $w \in F_n$  satisfies  $w = 1$  in  $G_\alpha$ ; the above construction fails because of the corresponding loops in the Cayley graph.

Let  $\alpha$  be an injective free group endomorphism. Let  $G_\alpha$  be the mapping-torus group of  $\alpha$ . Let  $\mathcal{C}(G_\alpha)$  be the Cayley complex of  $G_\alpha$  for the usual presentation  $G_\alpha = \langle x_1, \dots, x_n, t; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$ . By Lemma 13.3,  $\mathcal{C}(G_\alpha)$  is a mapping-telescope of a forest-map. We now want to see what happens with respect to metrics and dynamics. The Cayley graph of a group is equipped with a metric which makes each edge isometric to the interval  $(0, 1)$ . More generally, given a graph  $\Gamma$ , we call *standard metric*, and denote

by  $d_{\Gamma}^s$ , such a metric on  $\Gamma$ . We will call *mapping-telescope standard metric* any mapping-telescope  $d_{\Gamma}^s$ -metric on  $\mathcal{C}(G_{\alpha})$ .

LEMMA 13.5. *The mapping-torus group  $G_{\alpha}$  of an injective free group endomorphism acts cocompactly, properly discontinuously and isometrically on the Cayley complex  $\mathcal{C}(G_{\alpha})$  equipped with any mapping-telescope standard metric.*

*Proof.* We consider the usual action by left translations of the group on its Cayley graph. This action is extended in a natural way to a free action on the Cayley complex  $\mathcal{C}(G_{\alpha})$ . Let  $f$  denote the map giving the strata for the structure of forest-stack of  $\mathcal{C}(G_{\alpha})$ , see Lemma 13.3. For a mapping-telescope metric, all the strata  $f^{-1}(r)$  and  $f^{-1}(r+1)$  are isometric. And for a mapping-telescope standard metric all the strata  $f^{-1}(n)$ ,  $n \in \mathbf{Z}$ , are equipped with the standard metric. This readily implies that the above action is isometric.  $\square$

### 13.2 FREE GROUP ENDOMORPHISMS AND FOREST-MAPS

The main point of Lemma 13.6 below is the so-called ‘bounded-cancellation lemma’ of [7] for free group automorphisms, and of [10] for the injective free group endomorphisms.

LEMMA 13.6. *Let  $\alpha$  be an injective free group endomorphism. Let  $F$  and  $\tilde{\psi}$  be the forest and the forest-map on  $F$  given by Lemma 13.3. Then  $\tilde{\psi}$  is a weakly bi-Lipschitz forest-map of  $F$  equipped with the standard metric  $d_F^s$ .*

*Proof.* If  $w$  is any element in  $F_n = \langle x_1, \dots, x_n \rangle$ , and  $|\cdot|_{F_n}$  denotes the word-metric on  $F_n$ , then  $|\alpha(w)|_{F_n} \leq (\max_{i=1, \dots, n} |\alpha(x_i)|_{F_n}) |w|_{F_n}$ . By definition of the standard metric, and setting  $\mu_0 = \max_{i=1, \dots, n} |\alpha(x_i)|_{F_n}$ , the map  $\tilde{\psi}$  satisfies  $d_F^s(\tilde{\psi}(x), \tilde{\psi}(y)) \leq \mu_0 d_F^s(x, y)$  for any pair of vertices  $x, y$ . If  $x, y$  are not vertices, then they are joined in their stratum by a horizontal geodesic which is the concatenation of a path between two vertices, with two proper subsets of edges. By construction and simpliciality of  $\tilde{\psi}$ , proper subsets of edges are dilated by a bounded factor when applying  $\tilde{\psi}$ , so that the conclusion follows for the upper bound.

If  $w$  is any element in  $F_n$  then

$$|\alpha^{-1}(w)|_{F_n} \leq (\max_{i=1, \dots, n} |\alpha^{-1}(x_i)|_{F_n}) |w|_{F_n}.$$

Setting  $\mu_1 = \max_{i=1, \dots, n} |\alpha^{-1}(x_i)|_{F_n}$  we get  $|\alpha(w)|_{F_n} \geq \frac{1}{\mu_1} |w|_{F_n}$ . Therefore  $d_F^s(\tilde{\psi}(x), \tilde{\psi}(y)) \geq \frac{1}{\mu_1} d_F^s(x, y)$  for any pair of vertices  $x, y$ . The inequality