Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 49 (2003)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HYPERBOLICITY OF MAPPING-TORUS GROUPS AND SPACES

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DOI: https://doi.org/10.5169/seals-66690

DEFINITION 13.1 ([19], [3]). An injective endomorphism α of the rank n free group F_n is hyperbolic if there exist $\lambda_\alpha > 1$ and $j_\alpha > 0$ such that for any $w \in F_n$, either $\lambda_\alpha |w| \leq |\alpha^{j_\alpha}(w)|$ or w admits a preimage $\alpha^{-j_\alpha}(w)$ such that $\lambda_\alpha |w| \leq |\alpha^{-j_\alpha}(w)|$, where $|\,.\,|$ denotes the usual word-metric.

We recall that a subgroup H in a group G is *malnormal* if $w^{-1}Hw \cap H = \{1\}$ for any element $w \notin H$ of G. We state our theorem about mapping-torus groups as follows:

THEOREM 13.2. *Let α be an injective hyperbolic endomorphism of the rank n free group F_n. If the image of α is a malnormal subgroup of F_n then the mapping-torus group $G_\alpha = \langle x_1, \dots, x_n, t\,;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \dots, n \rangle$ is a hyperbolic group.*

13.1 RELATIONSHIPS WITH MAPPING-TELESCOPES

We consider the rank n free group $F_n = \langle x_1, \dots, x_n \rangle$. Let α be an injective endomorphism of F_n. Let $G_\alpha = \langle x_1, \dots, x_n, t\,;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \dots, n \rangle$ be the mapping-torus group of (α, F_n). We consider the Cayley graph Γ associated to the given system of generators. Let l be a loop in Γ whose associated word in the edges of Γ reads a relation $t^{-1}x_i t\alpha(x_i)^{-1}$. We attach a 2-cell by its boundary circle along any such loop l. The resulting topological space is a 2-complex. This is the Cayley complex of the mapping-torus group G_α for the given presentation.

Let us check that the above Cayley complex is a mapping-telescope of a forest-map. We consider the rose $\mathcal{R}_n$ with n petals. We label each edge by a generator x_i of F_n. We denote by ψ the simplicial map on $\mathcal{R}_n$ such that $\psi(x_i)$ is a locally injective path whose associated word in the edges of $\mathcal{R}_n$ reads $\alpha(x_i)$. Let us denote by T the universal covering of $\mathcal{R}_n$ (T is a tree) and by $\pi\colon T \to \mathcal{R}_n$ the associated covering-map. We denote by $\widehat{\psi}\colon T \to T$ a simplicial lift of ψ to T, that is $\pi \circ \widehat{\psi} = \psi \circ \pi$. We consider the mapping-torus of $(\psi, \mathcal{R}_n)$, i.e. the 2-complex $\mathcal{R}_n \times [0,1]/(x,1) \sim (\psi(x), 0)$. Then the universal covering of this mapping-torus is the mapping-telescope of $\widetilde{\psi}\colon F \to F$, where F and $\widetilde{\psi}$ are defined as follows:

- We denote by I the set of integers from 1 to $\mathrm{Card}(F_n/\operatorname{Im}(\alpha))$. The different classes are written $w_i \operatorname{Im}(\alpha)$, $i = 0, 1, \dots$. We denote by $\gamma\colon I \to \{w_0, w_1, \dots\}$ the bijection. Then the connected components of F are in bijection with $\mathbf{N}^{\mathrm{Card}(I)}$. Each connected component is the image, by a bijection μ, of a sequence of Card(I) integers. Each connected component $\mu(x_0, x_1, \dots)$ of F is homeomorphic to T via $\beta_{(x_0,x_1,\dots)}\colon \mu(x_0, x_1, \dots) \to T$.

• We define the restriction of $\widetilde{\psi}$ to any connected component $\mu((x_0, x_1, \dots))$ as follows:

If Card(I) $< +\infty$ then

$$\widetilde{\psi}|_{\mu((x_0,x_1,\dots))}\colon \begin{cases} \mu((x_0, x_1, \dots)) & \to & \mu((E[\frac{x_0}{\mathrm{Card}(I)}], x_1, \dots)) \\ x & \to & (\gamma(j)\beta^{-1}_{(x_0,x_1,\dots)}\widehat{\psi}\beta_{(x_0,x_1,\dots)})(x) \end{cases}$$

where $j < \mathrm{Card}(I)$ satisfies $E[\frac{x_0}{\mathrm{Card}(I)}] = k\,\mathrm{Card}(I) + j$.

If Card(I) $= +\infty$ then

$$\widetilde{\psi}|_{\mu((x_0,x_1,\dots))}\colon \begin{cases} \mu((x_0, x_1, \dots)) & \to & \mu((x_1, x_2, \dots)) \\ x & \to & (\gamma(x_0)\beta^{-1}_{(x_0,x_1,\dots)}\widehat{\psi}\beta_{(x_0,x_1,\dots)})(x) . \end{cases}$$

The mapping-torus of $(\psi, \mathcal{R}_n)$ is a 2-complex whose 1-skeleton is the rose with $n+1$ petals in bijection with $\{x_1, \dots, x_n, t\}$. There is one 2-cell for each relation $t^{-1}x_i t\alpha(x_i)^{-1}$. Thus the universal covering described above is the Cayley complex for G_α with the presentation $G_\alpha = \langle x_1, \dots, x_n, t\, ;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \dots, n\rangle$. We have thus proved

LEMMA 13.3. *Let α be an injective endomorphism of $F_n = \langle x_1, \dots, x_n\rangle$. Let $G_\alpha = \langle x_1, \dots, x_n, t\, ;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \dots, n\rangle$ be the mapping-torus group of α. Let $\mathcal{C}(G_\alpha)$ be the Cayley complex of G_α for the given presentation. Then $\mathcal{C}(G_\alpha)$ is the mapping-telescope of a forest-map.*

REMARK 13.4. If the endomorphism α is an automorphism then the above Cayley complex is the mapping-telescope of a tree-map. The tree is the universal covering of the rose with n petals. If the endomorphism α is not injective then some element $w \in F_n$ satisfies $w = 1$ in G_α; the above construction fails because of the corresponding loops in the Cayley graph.

Let α be an injective free group endomorphism. Let G_α be the mapping-torus group of α. Let $\mathcal{C}(G_\alpha)$ be the Cayley complex of G_α for the usual presentation $G_\alpha = \langle x_1, \dots, x_n, t\, ;\, t^{-1}x_i t = \alpha(x_i),\ i = 1, \dots, n\rangle$. By Lemma 13.3, $\mathcal{C}(G_\alpha)$ is a mapping-telescope of a forest-map. We now want to see what happens with respect to metrics and dynamics. The Cayley graph of a group is equipped with a metric which makes each edge isometric to the interval $(0, 1)$. More generally, given a graph Γ, we call *standard metric*, and denote by d^s_Γ, such a metric on Γ. We will call *mapping-telescope standard metric* any mapping-telescope d^s_Γ-metric on $\mathcal{C}(G_\alpha)$.

LEMMA 13.5. *The mapping-torus group G_α of an injective free group endomorphism acts cocompactly, properly discontinuously and isometrically on the Cayley complex $\mathcal{C}(G_\alpha)$ equipped with any mapping-telescope standard metric.*

Proof. We consider the usual action by left translations of the group on its Cayley graph. This action is extended in a natural way to a free action on the Cayley complex $\mathcal{C}(G_\alpha)$. Let f denote the map giving the strata for the structure of forest-stack of $\mathcal{C}(G_\alpha)$, see Lemma 13.3. For a mapping-telescope metric, all the strata $f^{-1}(r)$ and $f^{-1}(r+1)$ are isometric. And for a mapping-telescope standard metric all the strata $f^{-1}(n)$, $n \in \mathbf{Z}$, are equipped with the standard metric. This readily implies that the above action is isometric. □

13.2 FREE GROUP ENDOMORPHISMS AND FOREST-MAPS

The main point of Lemma 13.6 below is the so-called 'bounded-cancellation lemma' of [7] for free group automorphisms, and of [10] for the injective free group endomorphisms.

LEMMA 13.6. *Let α be an injective free group endomorphism. Let F and $\widetilde{\psi}$ be the forest and the forest-map on F given by Lemma 13.3. Then $\widetilde{\psi}$ is a weakly bi-Lipschitz forest-map of F equipped with the standard metric d^s_F.*

Proof. If w is any element in $F_n = \langle x_1, \dots, x_n\rangle$, and $|\,.\,|_{F_n}$ denotes the word-metric on F_n, then $|\alpha(w)|_{F_n} \leq (\max_{i=1,\dots,n} |\alpha(x_i)|_{F_n})|w|_{F_n}$. By definition of the standard metric, and setting $\mu_0 = \max_{i=1,\dots,n} |\alpha(x_i)|_{F_n}$, the map $\widetilde{\psi}$ satisfies $d^s_F(\widetilde{\psi}(x), \widetilde{\psi}(y)) \leq \mu_0 d^s_F(x,y)$ for any pair of *vertices* x, y. If x, y are not vertices, then they are joined in their stratum by a horizontal geodesic which is the concatenation of a path between two vertices, with two proper subsets of edges. By construction and simpliciality of $\widetilde{\psi}$, proper subsets of edges are dilated by a bounded factor when applying $\widetilde{\psi}$, so that the conclusion follows for the upper bound.

If w is any element in F_n then

$$\left|\alpha^{-1}(w)\right|_{F_n} \leq (\max_{i=1,\dots,n} \left|\alpha^{-1}(x_i)\right|_{F_n})|w|_{F_n}.$$

Setting $\mu_1 = \max_{i=1,\dots,n} \left|\alpha^{-1}(x_i)\right|_{F_n}$ we get $|\alpha(w)|_{F_n} \geq \frac{1}{\mu_1}|w|_{F_n}$. Therefore $d^s_F(\widetilde{\psi}(x), \widetilde{\psi}(y)) \geq \frac{1}{\mu_1} d^s_F(x,y)$ for any pair of *vertices* x, y. The inequality