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The L_{ij}, together with these isomorphisms, define a gerbe over $\mathrm{SU}(d+1)$, representing the generator of $H^3(\mathrm{SU}(d+1), \mathbf{Z})$.

More generally, consider any compact, simply connected, simple Lie group G of rank d. Up to conjugacy, G contains exactly $d+1$ elements with semi-simple centralizer. (For $G = \mathrm{SU}(d+1)$, these are the central elements.) Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1} \subset G$ be their conjugacy classes. We will define an invariant open cover $V_1, \ldots, V_{d+1}$ of G, with the property that each member of this cover admits an equivariant retraction onto the conjugacy class $\mathcal{C}_j \subset V_j$. It turns out that every semi-simple centralizer has a distinguished central extension by $\mathrm{U}(1)$. This central extension defines an equivariant bundle gerbe on $\mathcal{C}_j$, hence (by pull-back) an equivariant bundle gerbe over V_j. We will find that these gerbes over V_j glue together to produce a gerbe over G, using a gluing rule developed in this paper.

The organization of the paper is as follows. In Section 2 we review the theory of gerbes and pseudo-line bundles with connections, and discuss 'strong equivariance' under a group action. Section 4 describes gluing rules for bundle gerbes. Section 3 summarizes some facts about gerbes coming from central extensions. In Section 5 we give the construction of the basic gerbe over G outlined above, and in Section 6 we study the 'pre-quantization of conjugacy classes'.

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2. Gerbes with connections

In this section we review gerbes on manifolds, along the lines of Chatterjee-Hitchin and Murray.

2.1 Chatterjee-Hitchin gerbes

Let M be a manifold. Any Hermitian line bundle over M can be described by an open cover U_a, and transition functions $\chi_{ab}\colon U_a \cap U_b \to \mathrm{U}(1)$ satisfying a cocycle condition $(\delta\chi)_{abc} = \chi_{bc}\chi_{ac}^{-1}\chi_{ab} = 1$ on triple intersections. The cohomology class in $H^1(M,\underline{\mathrm{U}(1)}) = H^2(M,\mathbf{Z})$ defined by this cocycle is the Chern class of the line bundle. Chatterjee-Hitchin [10, 18, 17] suggested to realize classes in $H^3(M,\mathbf{Z})$ in a similar fashion, replacing $\mathrm{U}(1)$-valued functions with Hermitian line bundles. They define a gerbe to be a collection of Hermitian transition line bundles $L_{ab} \to U_a \cap U_b$ and a trivialization, i.e. unit length section, t_{abc} of the line bundle $(\delta L)_{abc} = L_{bc}L_{ac}^{-1}L_{ab}$ over triple intersections. These trivializations have to satisfy a compatibility relation over quadruple intersections,

$$(\delta t)_{abcd} \equiv t_{bcd}t_{acd}^{-1}t_{abd}t_{abc}^{-1} = 1\,,$$

which makes sense since $(\delta t)_{abcd}$ is a section of the *canonically* trivial bundle. (Each factor L_{ab} cancels with a factor L_{ab}^{-1}.) After passing to a refinement of the cover, such that all L_{ab} become trivializable, and picking trivializations, t_{abc} is simply a Čech cocycle of degree 2, hence defines a class in $H^2(M,\underline{\mathrm{U}(1)}) = H^3(M,\mathbf{Z})$. The class is independent of the choices made in this construction, and is called the *Dixmier-Douady class* of the gerbe.

Note that in practice, it is often not desirable to pass to a refinement. For example, if M is a connected, oriented 3-manifold, the generator of $H^3(M,\mathbf{Z}) = \mathbf{Z}$ can be described in terms of the cover U_1, U_2, where U_1 is an open ball around a given point $p \in M$, and $U_2 = M\backslash\{p\}$, using the degree one line bundle over $U_1 \cap U_2 \cong S^2 \times (0,1)$.

2.2 Bundle gerbes

Bundle gerbes were invented by Murray [24], generalizing the following construction of line bundles. Let $\pi\colon X \to M$ be a fiber bundle, or more generally a surjective submersion. (Different components of X may have different dimensions.) For each $k \geq 0$ let $X^{[k]}$ denote the k-fold fiber product of X with itself. There are $k+1$ projections $\partial^i\colon X^{[k+1]} \to X^{[k]}$, omitting the ith factor in the fiber product. Suppose we are given a smooth function $\chi\colon X^{[2]} \to \mathrm{U}(1)$, satisfying a cocycle condition $\delta\chi = 1$ where

$$\delta\chi := \partial_0^*\chi\partial_1^*\chi^{-1}\partial_2^*\chi\colon X^{[3]} \to \mathrm{U}(1)\,.$$

Then χ determines a Hermitian line bundle $L \to M$, with fibers at $m \in M$ the space of all linear maps $\phi\colon X_m = \pi^{-1}(m) \to \mathbf{C}$ such that $\phi(x) = \chi(x,x')\phi(x')$. Given local sections $\sigma_a\colon U_a \to X$ of X, the pull-backs of χ under the maps $(\sigma_a,\sigma_b)\colon U_a \cap U_b \to X^{[2]}$ give transition functions χ_{ab} for the line bundle.

Again, replacing $\mathrm{U}(1)$-valued functions by line bundles in this construction, one obtains a model for gerbes: A bundle gerbe is given by a line bundle $L \to X^{[2]}$ and a trivializing section t of the line bundle $\delta L = \partial_0^*L\otimes\partial_1^*L^{-1}\otimes\partial_2^*L$

over $X^{[3]}$, satisfying a compatibility condition $\delta t = 1$ over $X^{[4]}$ (which makes sense since δt is a section of the canonically trivial bundle $\delta\delta L$). Given local sections $\sigma_a\colon U_a \to X$, one can pull these data back under the maps $(\sigma_a, \sigma_b)\colon U_a \cap U_b \to X^{[2]}$ and $(\sigma_a, \sigma_b, \sigma_c)\colon U_a \cap U_b \cap U_c \to X^{[3]}$ to obtain a Chatterjee-Hitchin gerbe. The Dixmier-Douady class of (X, L, t) is by definition the Dixmier-Douady class of this Chatterjee-Hitchin gerbe; again this is independent of all choices. The Dixmier-Douady class behaves naturally under tensor product, pull-back and duals.

Notice that Chatterjee-Hitchin gerbes may be viewed as a special case of bundle gerbes, with X the disjoint union of the sets U_a in the given cover.

REMARK 2.1. In his original paper [24] Murray considered bundle gerbes only for fiber bundles, but this was found too restrictive. In [25], [29] the weaker condition (called 'locally split') is used that every point $x \in M$ admits an open neighborhood U and a map $\sigma\colon U \to X$ such that $\pi \circ \sigma = \mathrm{id}$. However, this condition seems insufficient in the smooth category, as the fiber product $X \times_M X$ need not be a manifold unless π is a submersion.

2.3 SIMPLICIAL GERBES

Murray's construction fits naturally into a wider context of *simplicial gerbes*. We refer to Mostow-Perchik's notes of lectures by R. Bott [23] and to Dupont's paper [12] for a nice introduction to simplicial manifolds, and to Stevenson [29] for their appearance in the gerbe context.

Recall that a *simplicial manifold* $M_\bullet$ is a sequence of manifolds $(M_n)_{n=0}^{\infty}$, together with *face maps* $\partial_i\colon M_n \to M_{n-1}$ for $i = 0, \ldots, n$ satisfying relations $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ for $i < j$. (The standard definition also involves *degeneracy maps* but these need not concern us here.) The *(fat) geometric realization* of $M_\bullet$ is the topological space $||M|| = \coprod_{n=1}^{\infty} \Delta^n \times M_n / \sim$, where Δ^n is the n-simplex and the relation is $(t, \partial_i(x)) \sim (\partial^i(t), x)$, for $\partial^i\colon \Delta^{n-1} \to \Delta^n$ the inclusion as the ith face. A (smooth) simplicial map between simplicial manifolds $M_\bullet, M'_\bullet$ is a collection of smooth maps $f_n\colon M_n \to M'_n$ intertwining the face maps; such a map induces a map between the geometric realizations.

EXAMPLES 2.2.

(a) If S is any manifold, one can define a simplicial manifold $E_\bullet S$ where $E_n S$ is the $n+1$-fold cartesian product of S, and ∂_j omits the jth factor. It is known [23] that the geometric realization $||ES||$ of this simplicial manifold is contractible. More generally, if $X \to M$ is a fiber bundle with fiber S, one can define a simplicial manifold $E_nX := X^{[n+1]}$, with face maps as in Section 2.2. The geometric realization $||EX||$ becomes a fiber bundle over M with contractible fiber $||ES||$.

(b) [22, 27] For any Lie group G there is a simplicial manifold $B_nG = G^n$. The face maps ∂_i for $0 < i < n$ are

$$\partial_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n),$$

while ∂_0 omits the first component and ∂_n the last component. The map $\pi_n\colon E_nG \to B_nG$ given by $\pi_n(k_0, \ldots, k_n) = (k_0k_1^{-1}, \ldots, k_{n-1}k_n^{-1})$ is simplicial, and the induced map on geometric realizations is a model for the classifying bundle $EG \to BG$.

(c) [27, 23] If $\mathcal{U} = \{U_a, a \in A\}$ is an open cover of M, one defines a simplicial manifold

$$\mathcal{U}_nM := \coprod_{(a_0,\ldots,a_n)\in A_n} U_{a_0\ldots a_n}$$

where A_n is the set of all sequences $(a_0, \ldots, a_n)$ such that $U_{a_0\ldots a_n} := U_{a_0} \cap \ldots \cap U_{a_n}$ is non-empty. The face maps are induced by the inclusions,

$$\partial_i\colon U_{a_0\ldots a_n} \hookrightarrow U_{a_0\ldots\widehat{a_i}\ldots a_n}.$$

One may view this as a special case of (a), with $X = \coprod_{a\in A} U_a$. It is known [23, Theorem 7.3] that $||\mathcal{U}M||$ is homotopy equivalent to M.

(d) [2] The definitions of E_nG and B_nG extend to Lie groupoids G over a base S. If $s, t\colon G \to S$ are the source and target maps, one defines E_nG as the $n+1$-fold fiber product of G with respect to the target map t. The space B_nG for $n \geq 1$ is the set of all $(g_1, \ldots, g_n) \in G^n$ with $s(g_j) = t(g_{j-1})$, while $B_0G = S$. The definition of the face maps $\partial_j\colon B_nG \to B_{n-1}G$ is as before for $n > 1$, while for $n = 1$, $\partial_0 = t$ and $\partial_1 = s$. We have a simplicial map $E_nG \to B_nG$ defined just as in the group case.

The bi-graded space of differential forms $\Omega^\bullet(M_\bullet)$ carries two commuting differentials d, δ, where d is the de Rham differential and $\delta\colon \Omega^k(M_n) \to \Omega^k(M_{n+1})$ is an alternating sum, $\delta\alpha = \sum_{i=0}^{n+1}(-1)^i\partial_i^*\alpha$. It is known [23, Theorem 4.2, Theorem 4.5] that the total cohomology of this double complex is the (singular) cohomology of the geometric realization, with coefficients in $\mathbf{R}$.

We will use the δ notation in many similar situations: For instance, given a Hermitian line bundle $L \to M_n$, we define a Hermitian line bundle $\delta L \to M_{n+1}$ as a tensor product,

$$\delta L = \partial_0^* L \otimes \partial_1^* L^{-1} \otimes \cdots \otimes \partial_{n+1}^* L^{\pm} .$$

The line bundle $\delta(\delta L) \to M_{n+1}$ is canonically trivial, due to the relations between face maps. If σ is a unitary section (i.e. a trivialization) of L, one uses a similar formula to define a unitary section $\delta\sigma$ of δL. Then $\delta(\delta\sigma) = 1$ (the identity section of the trivial line bundle $\delta(\delta L)$). For any unitary connection ∇ of L, one defines a unitary connection $\delta\nabla$ of δL in the obvious way.

CONVENTION. For the rest of this paper, we take all line bundles L to be *Hermitian* line bundles, and all connections ∇ on L to be *unitary* connections.

Let $M_\bullet$ be a simplicial manifold. One might define a simplicial line bundle as a collection of line bundles $L_n \to M_n$ such that the face maps $\partial_i \colon M_n \to M_{n-1}$ lift to line bundle homomorphisms $\hat{\partial}_i \colon L_n \to L_{n-1}$, satisfying the face map relations. Thus $L_\bullet$ is itself a simplicial manifold, and its geometric realization $||L||$ is a line bundle over $||M||$. Equivalently, the lifts $\hat{\partial}_i$ may be viewed as isomorphisms, $\partial_i^* L_{n-1} \to L_n$. In particular, we may identify L_n with the pull-back of $L := L_0$ under the nth-fold iterate $\partial_0 \circ \cdots \circ \partial_0$.

The isomorphisms $\partial_1^* L \cong \partial_0^* L = L_1$ determine a unitary section t of $\delta L \to M_1$, and the compatibility of isomorphisms

$$(\partial_0\partial_2)^* L \cong (\partial_0\partial_1)^* L \cong (\partial_0\partial_0)^* L = L_2$$

amount to the condition $\delta t = 1$. (Compatibility of the isomorphisms for L_n with $n \geq 3$ is then automatic.) That is, *a simplicial line bundle over $M_\bullet$ is given by a line bundle $L \to M_0$, together with a unitary section t of $\delta L \to M_1$, such that $\delta t = 1$ over M_2*. A unitary section s of L with $\delta s = t$ induces a unitary section of $||L|| \to ||M||$.

Taking L to be trivial, we see in particular that any $\mathrm{U}(1)$-valued function t on M_1, with $\delta t = 1$, defines a line bundle over the geometric realization. A trivialization of that line bundle is given by a $\mathrm{U}(1)$-valued function on M_0 satisfying $\delta s = t$. Replacing $\mathrm{U}(1)$-valued functions with line bundles, this motivates the following definition.

DEFINITION 2.3. A *simplicial gerbe* over $M_\bullet$ is a pair (L, t), consisting of a line bundle $L \to M_1$, together with a section t of $\delta L \to M_2$ satisfying $\delta t = 1$. A pseudo-line bundle for (L, t) is a pair (E, s), consisting of a line bundle $E \to M_0$ and a section s of $\delta E^{-1} \otimes L$ such that $\delta s = t$.

REMARK 2.4.

(a) We are using the notion of a simplicial gerbe only as a 'working definition'. It is clear from the discussion above that a more general notion would involve a gerbe over M_0.

(b) In [9], what we call simplicial gerbe is called a simplicial line bundle. The name pseudo-line bundle is adopted from [9], where it is used in a similar context.

A simplicial gerbe over $\mathcal{U}_\bullet M$ (for a cover $\mathcal{U}$ of M) is a Chatterjee-Hitchin gerbe, while a simplicial gerbe over $E_\bullet X = X^{[\bullet+1]}$ (for a surjective submersion $X \to M$) is a bundle gerbe. It is shown in [24] that the characteristic class of a bundle gerbe (X, L, t) vanishes if and only if it admits a pseudo-line bundle.

EXAMPLE 2.5 (Central extensions). (See [9, p. 615].) Let K be a Lie group. A simplicial line bundle over $B_\bullet K$ is the same thing as a group homomorphism $K \to \mathrm{U}(1)$: The line bundle $L \to B_0 K$ is trivial since $B_0 K$ is just a point, hence the unitary section t of δL becomes a $\mathrm{U}(1)$-valued function. The condition $\delta t = 1$ means that this function is a group homomorphism.

Similarly, a simplicial gerbe (Γ, τ) over $B_\bullet K$ is the same thing as a central extension

$$\mathrm{U}(1) \to \widehat{K} \to K.$$

Indeed, given the line bundle $\Gamma \to K$ let $\widehat{K}$ be the unit circle bundle inside Γ. The fiber of $\delta\Gamma \to K^2$ at (k_1, k_2) is a tensor product $\Gamma_{k_2}\Gamma^{-1}_{k_1k_2}\Gamma_{k_1}$, hence the section τ of $\delta\Gamma \to K^2$ defines a unitary isomorphism $\Gamma_{k_1}\Gamma_{k_2} \cong \Gamma_{k_1k_2}$, or equivalently a product on $\widehat{K}$ covering the group multiplication on K. Finally, the condition $\delta\tau = 1$ is equivalent to associativity of this product.

A pseudo-line bundle (E, s) for the simplicial gerbe (Γ, τ) is the same thing as a splitting of the central extension: Obviously E is trivial since $B_0 K$ is just a point; the section s defines a trivialization $\widehat{K} = K \times \mathrm{U}(1)$, and $\delta s = t$ means that this is a group homomorphism.

DEFINITION 2.6. A connection on a simplicial gerbe (L, t) over $M_\bullet$ is a line bundle connection ∇^L, together with a 2-form $B \in \Omega^2(M_0)$, such that $(\delta\nabla^L)\, t = 0$ and

$$\delta B = \frac{1}{2\pi i}\,\mathrm{curv}(\nabla^L).$$

Given a pseudo-line bundle $\mathcal{L} = (E, s)$, we say that ∇^E is a pseudo-line bundle connection if it has the property $((\delta\nabla^E)^{-1}\nabla^L)s = 0$.

Simplicial gerbes need not admit connections in general. A sufficient condition for the existence of a connection is that the δ-cohomology of the double complex $\Omega^k(M_n)$ vanishes in bidegrees $(1,2)$ and $(2,1)$. In particular, this holds true for bundle gerbes: Indeed it is shown in [24] that for any surjective submersion $\pi\colon X \to M$ the sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(X) \xrightarrow{\delta} \Omega^k(X^{[2]}) \xrightarrow{\delta} \Omega^k(X^{[3]}) \xrightarrow{\delta} \cdots \tag{2.1}$$

is exact, so the δ-cohomology vanishes in *all* degrees.

Thus, every bundle gerbe $\mathcal{G} = (X, L, t)$ over a manifold M (and in particular every Chatterjee-Hitchin gerbe) admits a connection. One defines the 3*-curvature* $\eta \in \Omega^3(M)$ of the bundle gerbe connection by $\pi^*\eta = \mathrm{d}B \in \ker\delta$. It can be shown that its cohomology class is the image of the Dixmier-Douady class $[\mathcal{G}]$ under the map $H^3(M,\mathbf{Z}) \to H^3(M,\mathbf{R})$. Similarly, if $\mathcal{G}$ admits a pseudo-line bundle $\mathcal{L} = (E,s)$, one can always choose a pseudo-line bundle connection ∇^E. The difference $\frac{1}{2\pi i}\,\mathrm{curv}(\nabla^E) - B$ is δ-closed and one defines the *error* 2*-form* of this connection by

$$\pi^*\omega = \frac{1}{2\pi i}\,\mathrm{curv}(\nabla^E) - B\,.$$

It is clear from the definition that $\mathrm{d}\omega + \eta = 0$.

REMARK 2.7. There is a notion of holonomy around surfaces for gerbe connections (cf. Hitchin [18] and Murray [24]), and in fact gerbe connections can be defined in terms of their holonomy (see Mackaay-Picken [20]).

2.4 EQUIVARIANT BUNDLE GERBES

Suppose G is a Lie group acting on X and on M, and that $\pi\colon X \to M$ is a G-equivariant surjective submersion. Then G acts on all fiber products $X^{[p]}$. We will say that a bundle gerbe $\mathcal{G} = (X, L, t)$ is G*-equivariant*, if L is a G-equivariant line bundle and t is a G-invariant section. An equivariant bundle gerbe defines a gerbe over the Borel construction[1]) $X_G = EG \times_G X \to M_G = EG \times_G M$, hence has an *equivariant* Dixmier-Douady class in $H^3(M_G,\mathbf{Z}) = H^3_G(M,\mathbf{Z})$. Similarly, we say that a pseudo-line bundle (E,s) for (X,L,t) is equivariant, provided E carries a G-action and s is an invariant section.

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[1]) We have not discussed bundle gerbes over infinite-dimensional spaces such as M_G. Recall however [4] that the classifying bundle $EG \to BG$ may be approximated by finite-dimensional principal bundles, and that equivariant cohomology groups of a given degree may be computed using such finite dimensional approximations.

REMARK 2.8. As pointed out in Mathai-Stevenson [21], this notion of equivariant bundle gerbe is sometimes 'really too strong': For instance, if $X = \coprod U_a$, for an open cover $\mathcal{U} = \{U_a, a \in A\}$, a G-action on X would amount to the cover being G-invariant. Brylinski [9] on the other hand gives a definition of equivariant Chatterjee-Hitchin gerbes that does not require invariance of the cover.

To define equivariant connections and curvature, we will need some notions from equivariant de Rham theory [15]. Recall that for a compact group G, the equivariant cohomology $H_G^\bullet(M, \mathbf{R})$ may be computed from Cartan's complex of equivariant differential forms $\Omega_G^\bullet(M)$, consisting of G-equivariant polynomial maps $\alpha\colon \mathfrak{g} \to \Omega(M)$. The grading is the sum of the differential form degree and twice the polynomial degree, and the differential reads

$$(\mathrm{d}_G\, \alpha)(\xi) = \mathrm{d}\, \alpha(\xi) - \iota(\xi_M)\alpha(\xi),$$

where $\xi_M = \frac{d}{dt}|_{t=0} \exp(-t\xi)$ is the generating vector field corresponding to $\xi \in \mathfrak{g}$. Given a G-equivariant connection ∇^L on an equivariant line bundle, one defines [3, Chapter 7] a d_G-closed equivariant curvature $\mathrm{curv}_G(\nabla^L) \in \Omega_G^2(M)$.

A equivariant connection on a G-equivariant bundle gerbe (X, L, t) over M is a pair (∇^L, B_G), where ∇^L is an invariant connection and $B_G \in \Omega_G^2(X)$ an equivariant 2-form, such that $\delta\nabla^L t = 0$ and $\delta B_G = \frac{1}{2\pi i}\mathrm{curv}_G(\nabla^L)$. Its equivariant 3-curvature $\eta_G \in \Omega_G^3(M)$ is defined by $\pi^*\eta_G = \mathrm{d}_G B_G$. Given an *invariant* pseudo-line bundle connection ∇^E on a equivariant pseudo-line bundle (E, s), one defines the equivariant error 2-form ω_G by

$$\pi^*\omega_G = \frac{1}{2\pi i}\mathrm{curv}_G(\nabla^E) - B_G\,.$$

Clearly, $\mathrm{d}_G\, \omega_G + \eta_G = 0$.

3. GERBES FROM PRINCIPAL BUNDLES

The following well-known example [7], [24] of a gerbe will be important for our construction of the basic gerbe over G. Suppose $\mathrm{U}(1) \to \widehat{K} \to K$ is a central extension, and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. Given a principal K-bundle $\pi\colon P \to B$, one constructs a bundle gerbe (P, L, t), sometimes called the lifting bundle gerbe. Observe that

$$E_n P = P \times_K E_n K\,,$$