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The L_{ij}, together with these isomorphisms, define a gerbe over $\mathrm{SU}(d+1)$, representing the generator of $H^3(\mathrm{SU}(d+1), \mathbf{Z})$.

More generally, consider any compact, simply connected, simple Lie group G of rank d. Up to conjugacy, G contains exactly $d+1$ elements with semi-simple centralizer. (For $G = \mathrm{SU}(d+1)$, these are the central elements.) Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1} \subset G$ be their conjugacy classes. We will define an invariant open cover $V_1, \ldots, V_{d+1}$ of G, with the property that each member of this cover admits an equivariant retraction onto the conjugacy class $\mathcal{C}_j \subset V_j$. It turns out that every semi-simple centralizer has a distinguished central extension by $\mathrm{U}(1)$. This central extension defines an equivariant bundle gerbe on $\mathcal{C}_j$, hence (by pull-back) an equivariant bundle gerbe over V_j. We will find that these gerbes over V_j glue together to produce a gerbe over G, using a gluing rule developed in this paper.

The organization of the paper is as follows. In Section 2 we review the theory of gerbes and pseudo-line bundles with connections, and discuss 'strong equivariance' under a group action. Section 4 describes gluing rules for bundle gerbes. Section 3 summarizes some facts about gerbes coming from central extensions. In Section 5 we give the construction of the basic gerbe over G outlined above, and in Section 6 we study the 'pre-quantization of conjugacy classes'.

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2. GERBES WITH CONNECTIONS

In this section we review gerbes on manifolds, along the lines of Chatterjee-Hitchin and Murray.

2.1 CHATTERJEE-HITCHIN GERBES

Let M be a manifold. Any Hermitian line bundle over M can be described by an open cover U_a, and transition functions $\chi_{ab}\colon U_a \cap U_b \to \mathrm{U}(1)$ satisfying a cocycle condition $(\delta\chi)_{abc} = \chi_{bc}\chi_{ac}^{-1}\chi_{ab} = 1$ on triple intersections. The cohomology class in $H^1(M, \underline{\mathrm{U}(1)}) = H^2(M, \mathbf{Z})$ defined by this cocycle is the Chern class of the line bundle. Chatterjee-Hitchin [10, 18, 17] suggested to realize classes in $H^3(M, \mathbf{Z})$ in a similar fashion, replacing $\mathrm{U}(1)$-valued functions with Hermitian line bundles. They define a gerbe to be a collection of Hermitian transition line bundles $L_{ab} \to U_a \cap U_b$ and a trivialization, i.e. unit length section, t_{abc} of the line bundle $(\delta L)_{abc} = L_{bc} L_{ac}^{-1} L_{ab}$ over triple intersections. These trivializations have to satisfy a compatibility relation over quadruple intersections,

$$(\delta t)_{abcd} \equiv t_{bcd} t_{acd}^{-1} t_{abd} t_{abc}^{-1} = 1\,,$$

which makes sense since $(\delta t)_{abcd}$ is a section of the *canonically* trivial bundle. (Each factor L_{ab} cancels with a factor L_{ab}^{-1}.) After passing to a refinement of the cover, such that all L_{ab} become trivializable, and picking trivializations, t_{abc} is simply a Čech cocycle of degree 2, hence defines a class in $H^2(M, \underline{\mathrm{U}(1)}) = H^3(M, \mathbf{Z})$. The class is independent of the choices made in this construction, and is called the *Dixmier-Douady class* of the gerbe.

Note that in practice, it is often not desirable to pass to a refinement. For example, if M is a connected, oriented 3-manifold, the generator of $H^3(M, \mathbf{Z}) = \mathbf{Z}$ can be described in terms of the cover U_1, U_2, where U_1 is an open ball around a given point $p \in M$, and $U_2 = M \backslash \{p\}$, using the degree one line bundle over $U_1 \cap U_2 \cong S^2 \times (0,1)$.

2.2 Bundle gerbes

Bundle gerbes were invented by Murray [24], generalizing the following construction of line bundles. Let $\pi\colon X \to M$ be a fiber bundle, or more generally a surjective submersion. (Different components of X may have different dimensions.) For each $k \geq 0$ let $X^{[k]}$ denote the k-fold fiber product of X with itself. There are $k+1$ projections $\partial^i\colon X^{[k+1]} \to X^{[k]}$, omitting the ith factor in the fiber product. Suppose we are given a smooth function $\chi\colon X^{[2]} \to \mathrm{U}(1)$, satisfying a cocycle condition $\delta\chi = 1$ where

$$\delta\chi := \partial_0^*\chi \partial_1^*\chi^{-1} \partial_2^*\chi\colon X^{[3]} \to \mathrm{U}(1)\,.$$

Then χ determines a Hermitian line bundle $L \to M$, with fibers at $m \in M$ the space of all linear maps $\phi\colon X_m = \pi^{-1}(m) \to \mathbf{C}$ such that $\phi(x) = \chi(x, x')\phi(x')$. Given local sections $\sigma_a\colon U_a \to X$ of X, the pull-backs of χ under the maps $(\sigma_a, \sigma_b)\colon U_a \cap U_b \to X^{[2]}$ give transition functions χ_{ab} for the line bundle.

Again, replacing $\mathrm{U}(1)$-valued functions by line bundles in this construction, one obtains a model for gerbes: A bundle gerbe is given by a line bundle $L \to X^{[2]}$ and a trivializing section t of the line bundle $\delta L = \partial_0^* L \otimes \partial_1^* L^{-1} \otimes \partial_2^* L$