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over $X^{[3]}$, satisfying a compatibility condition $\delta t = 1$ over $X^{[4]}$ (which makes sense since δt is a section of the canonically trivial bundle $\delta\delta L$). Given local sections $\sigma_a: U_a \rightarrow X$, one can pull these data back under the maps $(\sigma_a, \sigma_b): U_a \cap U_b \rightarrow X^{[2]}$ and $(\sigma_a, \sigma_b, \sigma_c): U_a \cap U_b \cap U_c \rightarrow X^{[3]}$ to obtain a Chatterjee-Hitchin gerbe. The Dixmier-Douady class of (X, L, t) is by definition the Dixmier-Douady class of this Chatterjee-Hitchin gerbe; again this is independent of all choices. The Dixmier-Douady class behaves naturally under tensor product, pull-back and duals.

Notice that Chatterjee-Hitchin gerbes may be viewed as a special case of bundle gerbes, with X the disjoint union of the sets U_a in the given cover.

REMARK 2.1. In his original paper [24] Murray considered bundle gerbes only for fiber bundles, but this was found too restrictive. In [25], [29] the weaker condition (called ‘locally split’) is used that every point $x \in M$ admits an open neighborhood U and a map $\sigma: U \rightarrow X$ such that $\pi \circ \sigma = \text{id}$. However, this condition seems insufficient in the smooth category, as the fiber product $X \times_M X$ need not be a manifold unless π is a submersion.

2.3 SIMPLICIAL GERBES

Murray’s construction fits naturally into a wider context of *simplicial gerbes*. We refer to Mostow-Perchik’s notes of lectures by R. Bott [23] and to Dupont’s paper [12] for a nice introduction to simplicial manifolds, and to Stevenson [29] for their appearance in the gerbe context.

Recall that a *simplicial manifold* M_\bullet is a sequence of manifolds $(M_n)_{n=0}^\infty$, together with *face maps* $\partial_i: M_n \rightarrow M_{n-1}$ for $i = 0, \dots, n$ satisfying relations $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ for $i < j$. (The standard definition also involves *degeneracy maps* but these need not concern us here.) The *(fat) geometric realization* of M_\bullet is the topological space $\|M\| = \coprod_{n=1}^\infty \Delta^n \times M_n / \sim$, where Δ^n is the n -simplex and the relation is $(t, \partial_i(x)) \sim (\partial^i(t), x)$, for $\partial^i: \Delta^{n-1} \rightarrow \Delta^n$ the inclusion as the i th face. A (smooth) simplicial map between simplicial manifolds M_\bullet, M'_\bullet is a collection of smooth maps $f_n: M_n \rightarrow M'_n$ intertwining the face maps; such a map induces a map between the geometric realizations.

EXAMPLES 2.2.

(a) If S is any manifold, one can define a simplicial manifold $E_\bullet S$ where $E_n S$ is the $n+1$ -fold cartesian product of S , and ∂_j omits the j th factor. It is known [23] that the geometric realization $\|ES\|$ of this simplicial manifold is contractible. More generally, if $X \rightarrow M$ is a fiber bundle with fiber S ,

one can define a simplicial manifold $E_n X := X^{[n+1]}$, with face maps as in Section 2.2. The geometric realization $\|EX\|$ becomes a fiber bundle over M with contractible fiber $\|ES\|$.

(b) [22, 27] For any Lie group G there is a simplicial manifold $B_n G = G^n$. The face maps ∂_i for $0 < i < n$ are

$$\partial_i(g_1, \dots, g_n) = (g_1, \dots, g_i g_{i+1}, \dots, g_n),$$

while ∂_0 omits the first component and ∂_n the last component. The map $\pi_n: E_n G \rightarrow B_n G$ given by $\pi_n(k_0, \dots, k_n) = (k_0 k_1^{-1}, \dots, k_{n-1} k_n^{-1})$ is simplicial, and the induced map on geometric realizations is a model for the classifying bundle $EG \rightarrow BG$.

(c) [27, 23] If $\mathcal{U} = \{U_a, a \in A\}$ is an open cover of M , one defines a simplicial manifold

$$\mathcal{U}_n M := \coprod_{(a_0, \dots, a_n) \in A_n} U_{a_0 \dots a_n}$$

where A_n is the set of all sequences (a_0, \dots, a_n) such that $U_{a_0 \dots a_n} := U_{a_0} \cap \dots \cap U_{a_n}$ is non-empty. The face maps are induced by the inclusions,

$$\partial_i: U_{a_0 \dots a_n} \hookrightarrow U_{a_0 \dots \widehat{a_i} \dots a_n}.$$

One may view this as a special case of (a), with $X = \coprod_{a \in A} U_a$. It is known [23, Theorem 7.3] that $\|\mathcal{U}M\|$ is homotopy equivalent to M .

(d) [2] The definitions of $E_n G$ and $B_n G$ extend to Lie groupoids G over a base S . If $s, t: G \rightarrow S$ are the source and target maps, one defines $E_n G$ as the $n+1$ -fold fiber product of G with respect to the target map t . The space $B_n G$ for $n \geq 1$ is the set of all $(g_1, \dots, g_n) \in G^n$ with $s(g_j) = t(g_{j-1})$, while $B_0 G = S$. The definition of the face maps $\partial_j: B_n G \rightarrow B_{n-1} G$ is as before for $n > 1$, while for $n = 1$, $\partial_0 = t$ and $\partial_1 = s$. We have a simplicial map $E_n G \rightarrow B_n G$ defined just as in the group case. \cdot

The bi-graded space of differential forms $\Omega^\bullet(M_\bullet)$ carries two commuting differentials d, δ , where d is the de Rham differential and $\delta: \Omega^k(M_n) \rightarrow \Omega^k(M_{n+1})$ is an alternating sum, $\delta\alpha = \sum_{i=0}^{n+1} (-1)^i \partial_i^* \alpha$. It is known [23, Theorem 4.2, Theorem 4.5] that the total cohomology of this double complex is the (singular) cohomology of the geometric realization, with coefficients in \mathbf{R} .

We will use the δ notation in many similar situations: For instance, given a Hermitian line bundle $L \rightarrow M_n$, we define a Hermitian line bundle $\delta L \rightarrow M_{n+1}$ as a tensor product,

$$\delta L = \partial_0^* L \otimes \partial_1^* L^{-1} \otimes \cdots \otimes \partial_{n+1}^* L^{\pm}.$$

The line bundle $\delta(\delta L) \rightarrow M_{n+1}$ is canonically trivial, due to the relations between face maps. If σ is a unitary section (i.e. a trivialization) of L , one uses a similar formula to define a unitary section $\delta\sigma$ of δL . Then $\delta(\delta\sigma) = 1$ (the identity section of the trivial line bundle $\delta(\delta L)$). For any unitary connection ∇ of L , one defines a unitary connection $\delta\nabla$ of δL in the obvious way.

CONVENTION. For the rest of this paper, we take all line bundles L to be *Hermitian* line bundles, and all connections ∇ on L to be *unitary* connections.

Let M_{\bullet} be a simplicial manifold. One might define a simplicial line bundle as a collection of line bundles $L_n \rightarrow M_n$ such that the face maps $\partial_i: M_n \rightarrow M_{n-1}$ lift to line bundle homomorphisms $\hat{\partial}_i: L_n \rightarrow L_{n-1}$, satisfying the face map relations. Thus L_{\bullet} is itself a simplicial manifold, and its geometric realization $\|L\|$ is a line bundle over $\|M\|$. Equivalently, the lifts $\hat{\partial}_i$ may be viewed as isomorphisms, $\partial_i^* L_{n-1} \rightarrow L_n$. In particular, we may identify L_n with the pull-back of $L := L_0$ under the n th-fold iterate $\partial_0 \circ \cdots \circ \partial_0$.

The isomorphisms $\partial_1^* L \cong \partial_0^* L = L_1$ determine a unitary section t of $\delta L \rightarrow M_1$, and the compatibility of isomorphisms

$$(\partial_0 \partial_2)^* L \cong (\partial_0 \partial_1)^* L \cong (\partial_0 \partial_0)^* L = L_2$$

amount to the condition $\delta t = 1$. (Compatibility of the isomorphisms for L_n with $n \geq 3$ is then automatic.) That is, a *simplicial line bundle over M_{\bullet}* is given by a line bundle $L \rightarrow M_0$, together with a unitary section t of $\delta L \rightarrow M_1$, such that $\delta t = 1$ over M_2 . A unitary section s of L with $\delta s = t$ induces a unitary section of $\|L\| \rightarrow \|M\|$.

Taking L to be trivial, we see in particular that any $U(1)$ -valued function t on M_1 , with $\delta t = 1$, defines a line bundle over the geometric realization. A trivialization of that line bundle is given by a $U(1)$ -valued function on M_0 satisfying $\delta s = t$. Replacing $U(1)$ -valued functions with line bundles, this motivates the following definition.

DEFINITION 2.3. A *simplicial gerbe over M_{\bullet}* is a pair (L, t) , consisting of a line bundle $L \rightarrow M_1$, together with a section t of $\delta L \rightarrow M_2$ satisfying $\delta t = 1$. A *pseudo-line bundle for (L, t)* is a pair (E, s) , consisting of a line bundle $E \rightarrow M_0$ and a section s of $\delta E^{-1} \otimes L$ such that $\delta s = t$.

REMARK 2.4.

(a) We are using the notion of a simplicial gerbe only as a 'working definition'. It is clear from the discussion above that a more general notion would involve a gerbe over M_0 .

(b) In [9], what we call simplicial gerbe is called a simplicial line bundle. The name pseudo-line bundle is adopted from [9], where it is used in a similar context.

A simplicial gerbe over $\mathcal{U} \bullet M$ (for a cover \mathcal{U} of M) is a Chatterjee-Hitchin gerbe, while a simplicial gerbe over $E \bullet X = X^{[\bullet+1]}$ (for a surjective submersion $X \rightarrow M$) is a bundle gerbe. It is shown in [24] that the characteristic class of a bundle gerbe (X, L, t) vanishes if and only if it admits a pseudo-line bundle.

EXAMPLE 2.5 (Central extensions). (See [9, p.615].) Let K be a Lie group. A simplicial line bundle over $B \bullet K$ is the same thing as a group homomorphism $K \rightarrow \mathrm{U}(1)$: The line bundle $L \rightarrow B_0 K$ is trivial since $B_0 K$ is just a point, hence the unitary section t of δL becomes a $\mathrm{U}(1)$ -valued function. The condition $\delta t = 1$ means that this function is a group homomorphism.

Similarly, a simplicial gerbe (Γ, τ) over $B \bullet K$ is the same thing as a central extension

$$\mathrm{U}(1) \rightarrow \widehat{K} \rightarrow K.$$

Indeed, given the line bundle $\Gamma \rightarrow K$ let \widehat{K} be the unit circle bundle inside Γ . The fiber of $\delta\Gamma \rightarrow K^2$ at (k_1, k_2) is a tensor product $\Gamma_{k_2} \Gamma_{k_1 k_2}^{-1} \Gamma_{k_1}$, hence the section τ of $\delta\Gamma \rightarrow K^2$ defines a unitary isomorphism $\Gamma_{k_1} \Gamma_{k_2} \cong \Gamma_{k_1 k_2}$, or equivalently a product on \widehat{K} covering the group multiplication on K . Finally, the condition $\delta\tau = 1$ is equivalent to associativity of this product.

A pseudo-line bundle (E, s) for the simplicial gerbe (Γ, τ) is the same thing as a splitting of the central extension: Obviously E is trivial since $B_0 K$ is just a point; the section s defines a trivialization $\widehat{K} = K \times \mathrm{U}(1)$, and $\delta s = t$ means that this is a group homomorphism.

DEFINITION 2.6. A connection on a simplicial gerbe (L, t) over $M \bullet$ is a line bundle connection ∇^L , together with a 2-form $B \in \Omega^2(M_0)$, such that $(\delta\nabla^L)t = 0$ and

$$\delta B = \frac{1}{2\pi i} \mathrm{curv}(\nabla^L).$$

Given a pseudo-line bundle $\mathcal{L} = (E, s)$, we say that ∇^E is a pseudo-line bundle connection if it has the property $((\delta\nabla^E)^{-1}\nabla^L)s = 0$.

Simplicial gerbes need not admit connections in general. A sufficient condition for the existence of a connection is that the δ -cohomology of the double complex $\Omega^k(M_n)$ vanishes in bidegrees $(1, 2)$ and $(2, 1)$. In particular, this holds true for bundle gerbes: Indeed it is shown in [24] that for any surjective submersion $\pi: X \rightarrow M$ the sequence

$$(2.1) \quad 0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(X) \xrightarrow{\delta} \Omega^k(X^{[2]}) \xrightarrow{\delta} \Omega^k(X^{[3]}) \xrightarrow{\delta} \dots$$

is exact, so the δ -cohomology vanishes in *all* degrees.

Thus, every bundle gerbe $\mathcal{G} = (X, L, t)$ over a manifold M (and in particular every Chatterjee-Hitchin gerbe) admits a connection. One defines the *3-curvature* $\eta \in \Omega^3(M)$ of the bundle gerbe connection by $\pi^*\eta = dB \in \ker \delta$. It can be shown that its cohomology class is the image of the Dixmier-Douady class $[\mathcal{G}]$ under the map $H^3(M, \mathbf{Z}) \rightarrow H^3(M, \mathbf{R})$. Similarly, if \mathcal{G} admits a pseudo-line bundle $\mathcal{L} = (E, s)$, one can always choose a pseudo-line bundle connection ∇^E . The difference $\frac{1}{2\pi i} \text{curv}(\nabla^E) - B$ is δ -closed and one defines the *error 2-form* of this connection by

$$\pi^*\omega = \frac{1}{2\pi i} \text{curv}(\nabla^E) - B.$$

It is clear from the definition that $d\omega + \eta = 0$.

REMARK 2.7. There is a notion of holonomy around surfaces for gerbe connections (cf. Hitchin [18] and Murray [24]), and in fact gerbe connections can be defined in terms of their holonomy (see Mackaay-Picken [20]).

2.4 EQUIVARIANT BUNDLE GERBES

Suppose G is a Lie group acting on X and on M , and that $\pi: X \rightarrow M$ is a G -equivariant surjective submersion. Then G acts on all fiber products $X^{[p]}$. We will say that a bundle gerbe $\mathcal{G} = (X, L, t)$ is *G -equivariant*, if L is a G -equivariant line bundle and t is a G -invariant section. An equivariant bundle gerbe defines a gerbe over the Borel construction¹⁾ $X_G = EG \times_G X \rightarrow M_G = EG \times_G M$, hence has an *equivariant* Dixmier-Douady class in $H^3(M_G, \mathbf{Z}) = H_G^3(M, \mathbf{Z})$. Similarly, we say that a pseudo-line bundle (E, s) for (X, L, t) is *equivariant*, provided E carries a G -action and s is an invariant section.

¹⁾ We have not discussed bundle gerbes over infinite-dimensional spaces such as M_G . Recall however [4] that the classifying bundle $EG \rightarrow BG$ may be approximated by finite-dimensional principal bundles, and that equivariant cohomology groups of a given degree may be computed using such finite dimensional approximations.