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All of these constructions can be made equivariant in a rather obvious way: Thus if G is another Lie group and P is a G-invariant principal K-bundle, any $\varrho \in \mathrm{Hom}(\pi_1(K), \mathrm{U}(1))$ defines a G-equivariant bundle gerbe (P, L, t) (with flat connection) over M. If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, there is a G-equivariant pseudo-line bundle for this gerbe. Furthermore any choice of G-equivariant principal connection on P defines a G-equivariant pseudo-line bundle connection, with equivariant error 2-form $\pi^*\omega_G = \langle \mu, F_G^\theta \rangle$ where $F_G^\theta \in \Omega_G^2(P, \mathfrak{k})$ is the equivariant curvature.

4. Gluing data

In this Section we describe a procedure for gluing a collection of bundle gerbes (X_i, L_i, t_i) on open subsets $V_i \subset M$, with pseudo-line bundles of their quotients on overlaps[2]). We begin with the somewhat simpler case that the surjective submersions $X_i \to V_i$ are obtained by restricting a surjective submersion $X \to M$, and later reduce the general case to this special case.

Thus, let $\pi \colon X \to M$ be a surjective submersion and let V_i, $i = 0, \dots, d$ an open cover of M. Let $X_i = X|_{V_i}$, and more generally $X_I = X|_{V_I}$ where V_I is the intersection of all V_i with $i \in I$.

Suppose we are given bundle gerbes (X_i, L_i, t_i) over V_i and pseudo-line bundles (E_{ij}, s_{ij}) for the quotients $(X_{ij}, L_j L_i^{-1}, t_j t_i^{-1})$ over $V_i \cap V_j$, where $E_{ij} = E_{ji}^{-1}$ and $s_{ij} = s_{ji}^{-1}$. Note that $E_{ij} E_{jk} E_{ki}$ is a pseudo-line bundle for the trivial gerbe, hence is a pull-back $\pi^* F_{ijk}$ of a line bundle $F_{ijk} \to M$, and we will also require a unitary section u_{ijk} of that line bundle. Under suitable conditions the data (E_{ij}, s_{ij}) and u_{ijk} can be used to 'glue' the gerbes (X_i, L_i, t_i). The glued gerbe will be defined over the disjoint union $\coprod_{i=1}^d X_i$. We have

$$\Big(\coprod_{i=1}^{d} X_i\Big)^{[2]} = \coprod_{ij} X_i \times_M X_j$$

$$\Big(\coprod_{i=1}^{d} X_i\Big)^{[3]} = \coprod_{ijk} X_i \times_M X_j \times_M X_k$$

$$\dots$$

Hence, the glued gerbe will be of the form $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$ where L_{ij} are line bundles over $X_i \times_M X_j$ and t_{ijk} unitary sections of a line bundle $(\delta L)_{ijk}$

[2]) See Stevenson [29] for similar gluing constructions.

over $\coprod_{ijk} X_i \times_M X_j \times_M X_k$. We will define L_{ij} by tensoring $L_i \to X^{[2]}$ (restricted to $X_i \times_M X_j$) with the pull-back of E_{ij} under the map $\partial_1 \colon X_i \times_M X_j \to X_{ij}$.

PROPOSITION 4.1. *Suppose the sections* u_{ijk} *satisfy the cocycle condition* $u_{jkl}u_{ikl}^{-1}u_{ijl}u_{ijk}^{-1} = 1$, *and the sections* s_{ij} *satisfy a cocycle condition* $s_{ij}s_{jk}s_{ki} = 1$. *Then there is a well-defined gerbe* $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$ *over* M, *where* $L_{ij} \to X_i \times_M X_j$ *is the line bundle*

$$L_{ij} = L_j \otimes \partial_1^* E_{ij}$$

and t_{ijk} *is a section of* $(\delta L)_{ijk} \to X_i \times_M X_j \times_M X_k$ *given by*

$$t_{ijk} = t_k \otimes \partial_2^* s_{kj} \otimes \partial_2^* \partial_1^* \pi^* u_{ijk}\,. \tag{4.1}$$

Proof. A short calculation gives

$$(\delta L)_{ijk} = (\delta L_k) \otimes \partial_2^*(L_j L_k^{-1} \delta E_{kj}^{-1}) \otimes \partial_2^* \partial_1^* \pi^* F_{ijk}\,,$$

showing that t_{ijk} is a well-defined section of $(\delta L)_{ijk}$. One finds furthermore

$$\begin{aligned}(\delta t)_{ijkl} &= (\delta t_l) \otimes \partial_3^*\left(t_l t_k^{-1} \delta s_{kl}^{-1} \otimes \partial_2^*\left(s_{lj}s_{jk}s_{kl} \otimes \partial_1^* \pi^*(u_{jkl}u_{ikl}^{-1}u_{ijl}u_{ijk}^{-1})\right)\right)\\ &= \partial_3^* \partial_2^*\left(s_{lj}s_{jk}s_{kl} \otimes \partial_1^* \pi^*(u_{jkl}u_{ikl}^{-1}u_{ijl}u_{ijk}^{-1})\right)\end{aligned}$$

which equals 1 under the given assumptions on u and s.

The gluing construction described in this Proposition is particularly natural for Chatterjee-Hitchin gerbes: Suppose $\mathcal{U}$ is an open cover of M, and $X = \coprod_{U \in \mathcal{U}} U$. For any decomposition $\mathcal{U} = \coprod_{i=1}^d \mathcal{U}_i$ let $V_i = \cup_{U \in \mathcal{U}_i} U$, and $X_i = \coprod_{U \in \mathcal{U}_i} U$. Note that in this case,

$$\coprod_i X_i = X\,.$$

Suppose (L_i, t_i) are Chatterjee-Hitchin gerbes for the cover $\mathcal{U}_i$ of V_i, and that we are given pseudo-line bundles (E_{ij}, s_{ij}) and a section u_{ijk} as above. Note that the E_{ij} are a collection of line bundles over intersections $U_a \cap U_b$ where $U_a \in \mathcal{U}_i$ and $U_b \in \mathcal{U}_j$. The gluing construction gives a Chatterjee-Hitchin gerbe (L, t) for the cover $\mathcal{U}$ of M, where the E_{ij} enter the definition of transition line bundles between open sets in distinct $\mathcal{U}_i$, $\mathcal{U}_j$.

REMARK 4.2. Suppose $X = M$, and that all L_i, t_i, s_{ij} are trivial. Then the gerbe described in Proposition 4.1 *is* a Chatterjee-Hitchin gerbe for the cover $\{V_i\}$. The E_{ij} now play the role of transition line bundles, and u_{ijk} play the role of t.

Suppose now that, in addition to the assumptions of Proposition 4.1, we have gerbe connections (∇^{L_i}, B_i) and pseudo-line bundle connections $\nabla^{E_{ij}} = (\nabla^{E_{ji}})^{-1}$. Let ω_{ij} denote the error 2-form for $\nabla^{E_{ij}}$.

PROPOSITION 4.3. *The connections* $\nabla^{L_{ij}} = \nabla^{L_j} \otimes \partial_1^* \nabla^{E_{ij}}$ *on* L_{ij}, *together with the two forms* $B_i \in \Omega^2(X_i)$, *define a gerbe connection if all error 2-forms* ω_{ij} *vanish, and if*

$$\nabla^{E_{ij}} \nabla^{E_{jk}} \nabla^{E_{ki}} (\pi^* u_{ijk}) = 0\,.$$

Proof. Let B be the 2-form on $\coprod X_i$ given by B_i on X_i. We first verify that $\frac{1}{2\pi i}\,\mathrm{curv}(\nabla^{L_{ij}}) = (\delta B)_{ij}$:

$$\begin{aligned}\frac{1}{2\pi i}\,\mathrm{curv}(\nabla^{L_{ij}}) &= \frac{1}{2\pi i}\,\mathrm{curv}(\nabla^{L_j}) + \frac{1}{2\pi i}\partial_1^*\,\mathrm{curv}(\nabla^{E_{ij}})\\ &= \delta B_j + \partial_1^*(B^j - B^i + \pi^*\omega_{ij})\\ &= \partial_0^* B_j - \partial_1^* B_i = (\delta B)_{ij}\,.\end{aligned}$$

Next, we check that t_{ijk} is parallel for $(\delta\nabla^L)_{ijk}$:

$$\begin{aligned}(\delta\nabla^L)_{ijk} &= \partial_0^*\nabla^{L_{jk}} \partial_1^*(\nabla^{L_{ik}})^{-1} \partial_2^*\nabla^{L_{ij}}\\ &= \delta\nabla^{L_k} \otimes \partial_2^*(\nabla^{L_k}(\nabla^{L_j})^{-1}\delta\nabla^{E_{jk}}) \otimes \partial_2^*\partial_1^*(\nabla^{E_{ij}}\nabla^{E_{jk}}\nabla^{E_{ki}})\,.\end{aligned}$$

This annihilates (4.1) as required.

We now describe a slightly more complicated gluing construction, in which the X_i are not simply the restrictions of a surjective submersion $X \to M$. Instead, we assume that for each I we are given a surjective submersion $\pi_I \colon X_I \to V_I$ are surjective submersions, and for each $I \supset J$ a fiber preserving smooth map $f_I^J \colon X_I \to X_J$, with the compatibility condition $f_J^K \circ f_I^J = f_I^K$ for $I \supset J \supset K$. Our gluing data will consist of the following:

(i) Over each V_i, bundle gerbes (X_i, L_i, t_i) with connections (∇^{L_i}, B_i).

(ii) Over each V_{ij}, pseudo-line bundles $E_{ij} = E_{ji}^{-1}, s_{ij} = s_{ji}^{-1}$ with connections $\nabla^{E_{ij}} = (\nabla^{E_{ji}})^{-1}$ for the bundle gerbe (X_{ij}, L_{ij}, t_{ij}), given as the quotient of the pull-back of (X_j, L_j, t_j) by f_{ij}^j and the pull-back of (X_i, L_i, t_i) by f_{ij}^i.

(iii) Over triple intersections, unitary sections u_{ijk} of the line bundle $F_{ijk} \to V_{ijk}$ defined by tensoring the pull-backs of E_{ij}, E_{jk}, E_{ki} by the maps f_{ijk}^{ij}, f_{ijk}^{jk}, f_{ijk}^{ki}.

We require that the s_{ij} and u_{ijk} satisfy a cocycle condition similar to Proposition 4.1, that all error 2-forms ω_{ij} are zero, and that the connections $\nabla^{E_{ij}}$ satisfy a compatibility condition as in 4.3.

These data may be used to define a bundle gerbe over M, by reducing to the setting of Propositions 4.1, 4.3. As a first step we construct a more convenient cover.

LEMMA 4.4. *There are open subsets U_I of M, with $\overline{U}_I \subset V_I$, and $\bigcup_I U_I = M$, such that*

$$\overline{U}_I \cap \overline{U}_J = \emptyset \quad \textit{unless} \quad J \subset I \textit{ or } I \subset J\,.$$

The collection of open subsets

$$V_i' = M \backslash \bigcup_{J \not\ni i} \overline{U}_J$$

is a shrinking of the open cover V_i, that is, $\bigcup V_i' = M$ and $\overline{V_i'} \subset V_i$.

The proof of this technical lemma is deferred to Appendix A. Now set $X = \coprod_I X_I|_{U_I}$. By definition of V_i', the restriction $X_i' = X|_{V_i'}$ is given by

$$X_i' = \coprod_{J \ni i} X_J|_{U_J \cap V_i'}\,.$$

More generally, letting $V_I' = \bigcap_{i \in I} V_i'$ and $X_I' = X|_{V_I'}$ we have

$$X_I' = \coprod_{J \supset I} X_J|_{U_J \cap V_I'}\,.$$

Let $X_I' \to X_I|_{V_I'}$ be the fiber preserving map, given on $X_J|_{U_J \cap V_I'}$ by the map $f_J^I \colon X_J \to X_I$. Using these maps, we can pull-back our gluing data: Let (X_i', L_i', t_i') be the pull-back of the bundle gerbe (X_i, L_i, t_i) under the map $X_i' \to X_i$, equipped with the pull-back connection. On overlaps V_{ij}', we let (E_{ij}', s_{ij}') be the pseudo-line bundle with connections defined by pulling back (E_{ij}, s_{ij}). The gluing data obtained in this way satisfy the conditions from Propositions 4.1 and 4.3, and hence give rise to a bundle gerbe with connection over M.

REMARK 4.5. In our applications, the line bundles E_{ij} are in fact trivial, so one can simply take $u_{ijk} = 1$ in terms of the trivialization. The s_{ij} are U(1)-valued functions in this case, and the compatibility condition reads $s_{ij}s_{jk}s_{ki} = 1$ over X_{ijk}.

The gluing constructions generalize equivariant bundle gerbes in a straightforward way.