5. The basic gerbe over a compact simple Lie group

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

5. The basic gerbe over a compact simple Lie group

In this section we explain our construction of the basic gerbe over a compact, simple, simply connected Lie group.

5.1 Notation

Let G be a compact, simple, simply connected Lie group, with Lie algebra $\mathfrak{g}$. For any action of $G \times M \to M$, $(g,m) \mapsto g.m$ on a manifold M, we will denote by G_m the stabilizer group of a point $m \in M$. If $M = G$ or $M = \mathfrak{g}$, we will always consider the adjoint action of G unless specified otherwise. For instance, G_g for denotes the centralizer of an element $g \in G$.

Choose a maximal torus T of G, with Lie algebra $\mathfrak{t}$. Let $\Lambda = \ker(\exp|_{\mathfrak{t}})$ be the integral lattice and $\Lambda^* \subset \mathfrak{t}^*$ its dual, the (real) weight lattice. Equivalently, Λ is characterized as the lattice generated by the coroots $\check{\alpha}$ for the (real) roots α. Recall that the *basic inner product* $\cdot$ on $\mathfrak{g}$ is the unique invariant inner product such that $\check{\alpha} \cdot \check{\alpha} = 2$ for all long roots α. Throughout this paper, we will use the basic inner product to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Choose a collection of simple roots $\alpha_1, \ldots, \alpha_d \in \Lambda^*$ and let $\mathfrak{t}_+ = \{\xi \mid \alpha_j \cdot \xi \geq 0,\ j = 1, \ldots, d\}$ be the corresponding positive Weyl chamber. The fundamental alcove $\mathfrak{A}$ is the subset cut out from $\mathfrak{t}_+$ by the additional inequality $\alpha_0 \cdot \xi \geq -1$ where α_0 is the lowest root.

The fundamental alcove parametrizes conjugacy classes in G, in the sense that each conjugacy class contains a unique point $\exp\xi$ with $\xi \in \mathfrak{A}$. The quotient map will be denoted $q\colon G \to \mathfrak{A}$. Let $\mu_0, \ldots, \mu_d$ be the vertices of $\mathfrak{A}$, with $\mu_0 = 0$. For any $I \subseteq \{0, \ldots, d\}$, all group elements $\exp\xi$ with ξ in the open face spanned by μ_j with $j \in I$ have the same centralizer, denoted G_I. In particular, G_j will denote the centralizer of $\exp\mu_j$.

For each j let $\mathfrak{A}_j \subset \mathfrak{A}$ be the open star at μ_j, i.e. the union of all open faces containing μ_j in their closure. Put differently, $\mathfrak{A}_j$ is the complement of the closed face opposite to the vertex μ_j. We will work with the open cover of G given by the pre-images, $V_j = q^{-1}(\mathfrak{A}_j)$. More generally let $\mathfrak{A}_I = \bigcap_{j\in I}\mathfrak{A}_j$, and $V_I := q^{-1}(\mathfrak{A}_I)$. The flow-out $S_I = G_I . \exp(\mathfrak{A}_I)$ of $\exp(\mathfrak{A}_I) \subset T$ under the action of G_I is an open subset of G_I, and is a slice for the conjugation action of G. That is,

$$G \times_{G_I} S_I = V_I\,.$$

We let $\pi_I\colon V_I \to G/G_I$ denote the projection to the base.

5.2 The basic 3-form on G

Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left- and right-invariant Maurer-Cartan forms on G, respectively. The 3-form $\eta \in \Omega^3(G)$ given by[3])

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] = \frac{1}{12}\theta^R \cdot [\theta^R, \theta^R]$$

is closed, and has a closed equivariant extension $\eta_G \in \Omega^3_G(G)$ given by

$$\eta_G(\xi) := \eta - \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi .$$

Their cohomology classes represent generators of $H^3(G, \mathbf{Z}) = \mathbf{Z}$ and $H^3_G(G, \mathbf{Z}) = \mathbf{Z}$, respectively. The pull-back of η_G to any conjugacy class $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow G$ is exact. In fact, let $\omega_{\mathcal{C}} \in \Omega^2(\mathcal{C})^G \subset \Omega^2_G(\mathcal{C})$ be the invariant 2-form given on generating vector fields $\xi_{\mathcal{C}}, \xi'_{\mathcal{C}}$ for $\xi, \xi' \in \mathfrak{g}$ by the formula

$$\omega_{\mathcal{C}}(\xi_{\mathcal{C}}(g), \xi'_{\mathcal{C}}(g)) = \tfrac{1}{2}\xi \cdot (\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi' .$$

Then [1, 16]

$$\mathrm{d}_G\, \omega_{\mathcal{C}} + \iota^*_{\mathcal{C}} \eta_G = 0 .$$

We will now show that η_G is exact over each of the open subsets V_j. Let $\mathcal{C}_j = q^{-1}(\mu_j) \subset V_j$ be the conjugacy classes corresponding to the vertices.

Lemma 5.1. *The linear retraction*

$$[0,1] \times \mathfrak{A}_j \to \mathfrak{A}_j, \quad (t, \mu_j + \zeta) \mapsto \mu_j + (1-t)\,\zeta$$

of $\mathfrak{A}_j$ onto the vertex μ_j lifts uniquely to a smooth G-equivariant retraction from V_j onto $\mathcal{C}_j$.

Proof. Recall that the slice S_j is an open neighborhood of $\exp(\mu_j)$ in G_j. Any G_j-equivariant retraction from S_j onto $\exp \mu_j$ extends uniquely to a G-equivariant retraction from $V_j = G \times_{G_j} S_j$ onto $\mathcal{C}_j$. Note that $S'_j = G_j . (\mathfrak{A}_j - \mu_j)$ is a star-shaped open neighborhood of 0 in $\mathfrak{g}_j$, and that $S'_j \to S_j$, $\zeta \mapsto \exp(\mu_j)\exp(\zeta)$ is a G_j-equivariant diffeomorphism. The linear retraction of S'_j onto the origin gives the desired retraction of S_j. Uniqueness is clear, since the retraction has to preserve $\exp(\mathfrak{A}_j) \subset V_j$, by equivariance.

[3]) For $\mathfrak{g}$-valued forms β_1, β_2, the bracket $[\beta_1, \beta_2]$ denotes the $\mathfrak{g}$-valued form obtained by applying the Lie bracket $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ to the $\mathfrak{g} \otimes \mathfrak{g}$-valued form $\beta_1 \wedge \beta_2$.

Let

$$\mathbf{h}_j\colon \Omega^p(V_j) \to \Omega^p([0,1]\times V_j) \to \Omega^{p-1}(V_j)$$

be the de Rham homotopy operator for this retraction, given (up to a sign) by pull-back under the retraction, followed by integration over the fibers of $[0,1]\times V_j \to V_j$. It has the property

$$\mathrm{d}_G\,\mathbf{h}_j + \mathbf{h}_j\,\mathrm{d}_G = \mathrm{Id} - \pi_j^*\iota_j^* \tag{5.1}$$

where $\iota_j\colon \mathcal{C}_j \to V_j$ is the inclusion and $\pi_j\colon V_j = G\times_{G_j} S_j \to G/G_j = \mathcal{C}_j$ the projection. Let $(\varpi_j)_G = \mathbf{h}_j\eta_G - \pi_j^*\omega_{\mathcal{C}_j} \in \Omega^2_G(V_j)$, and write $(\varpi_j)_G = \varpi_j - \Psi_j$ where $\varpi_j \in \Omega^2(V_j)$ and $\Psi_j \in \Omega^0(V_j, \mathfrak{g})$.

PROPOSITION 5.2. *The equivariant* 2*-form* $(\varpi_j)_G = \varpi_j - \Psi_j$ *has the following properties.*

(a) $\mathrm{d}_G(\varpi_j)_G = \eta_G$.

(b) *The pull-back of* $(\varpi_j)_G$ *to a conjugacy class* $\mathcal{C}\subset V_j$ *is given by*

$$\iota_{\mathcal{C}}^*(\varpi_j)_G = \Psi_j^*(\omega_{\mathcal{O}})_G - \omega_{\mathcal{C}}\,,$$

where $(\omega_{\mathcal{O}})_G$ *is the equivariant symplectic form on the adjoint orbit* $\mathcal{O} = \Psi_j(\mathcal{C})$,

(c) *The pull-back of* Ψ_j *to the conjugacy class* $\mathcal{C}_j$ *vanishes. In fact,* $\Psi_j(\exp\xi) = \xi - \mu_j$ *for all* $\xi \in \mathfrak{A}_j$.

(d) *Over each intersection* $V_{ij} = V_i\cap V_j$, *the difference* $\Psi_i - \Psi_j$ *takes values in the adjoint orbit* $\mathcal{O}_{ij}$ *through* $\mu_j - \mu_i \in \mathfrak{g}\cong\mathfrak{g}^*$. *Furthermore,*

$$(\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G$$

where $p_{ij}\colon V_{ij}\to\mathcal{O}_{ij}$ *is the map defined by* $\Psi_i - \Psi_j$, *and* $(\omega_{\mathcal{O}_{ij}})_G$ *is the equivariant symplectic form on the orbit.*

Proof. (a) holds by construction. (b) follows from the observation that $\iota_{\mathcal{C}}^*(\varpi_j)_G + \omega_{\mathcal{C}}$ is an equivariantly closed 2-form on $\mathcal{C}_j$, with Ψ_j as its moment map. To prove (c) we note that since the retraction is equivariant, we have $\tilde{\mathbf{h}}_j\circ(\exp|_{\mathfrak{A}_j})^* = (\exp|_{\mathfrak{A}_j})^*\circ\mathbf{h}_j$ where $(\exp|_{\mathfrak{A}_j})^*$ is pull-back to $\mathfrak{A}_j\subset\mathfrak{t}$ and where $\tilde{\mathbf{h}}_j$ is the homotopy operator for the linear retraction of $\mathfrak{t}$ onto $\{\mu_j\}$. Let $\nu\colon\mathfrak{A}_j\to\mathfrak{t}$ be the coordinate function (inclusion). Then

$$\tilde{\mathbf{h}}_j\circ(\exp|_{\mathfrak{A}_j})^*\tfrac{1}{2}(\theta^L+\theta^R) = \tilde{\mathbf{h}}_j\circ\mathrm{d}\nu = \nu - \mu_j\,,$$

proving that $(\exp|_{\mathfrak{A}_j})^*\Psi_j = \nu - \mu_j$. This yields (c), by equivariance. For $\nu\in\mathfrak{A}_{ij}$ we have, using (c),

$$(\Psi_i - \Psi_j)(\exp \nu) = (\nu - \mu_i) - (\nu - \mu_j) = \mu_j - \mu_i\,.$$

By equivariance, it follows that $\Psi_i - \Psi_j$ takes values in the adjoint orbit through $\mu_j - \mu_i$. The difference $\varpi_i - \varpi_j$ vanishes on the maximal torus T, and is therefore determined by its contractions with generating vector fields. Since $\Psi_i - \Psi_j$ is a moment map for $\varpi_i - \varpi_j$, it follows that $\varpi_i - \varpi_j$ equals the pull-back of the symplectic form on $G\,.(\mu_j - \mu_i)$.

5.3 The special unitary group

For the special unitary group $G = \mathrm{SU}(d+1)$, the construction of the basic gerbe simplifies due to the fact that in this case all vertices μ_j of the alcove are contained in the weight lattice. In fact the gerbe is presented as a Chatterjee-Hitchin gerbe for the cover $\mathcal{V} = \{V_i,\ i = 0, \ldots, d\}$.

For each weight $\mu \in \Lambda^* \subset \mathfrak{t} \subset \mathfrak{g}$, let G_μ be its stabilizer for the adjoint action and let $\mathbf{C}_\mu$ the 1-dimensional G_μ-representation with infinitesimal character μ. Let the line bundle $L_\mu = G \times_{G_\mu} \mathbf{C}_\mu$ equipped with the unique left-invariant connection ∇. Then L_μ is a G-equivariant pre-quantum line bundle for the orbit $\mathcal{O} = G\,.\,\mu$. That is,

$$\frac{i}{2\pi}\,\mathrm{curv}_G(\nabla) = (\omega_{\mathcal{O}})_G := \omega_{\mathcal{O}} - \Phi_{\mathcal{O}}$$

where $\omega_{\mathcal{O}}$ is the symplectic form and $\Phi_{\mathcal{O}} \colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is the moment map given as inclusion.

In particular, in the case of $\mathrm{SU}(d+1)$ all orbits $\mathcal{O}_{ij} = G\,.(\mu_j - \mu_i)$ carry G-equivariant pre-quantum line bundles. Recall the fibrations $p_{ij} \colon V_{ij} \to \mathcal{O}_{ij}$ defined by $\Psi_i - \Psi_j$, and let

$$L_{ij} = p_{ij}^*\,(L_{\mu_j - \mu_i})\,,$$

equipped with the pull-back connection. For any triple intersection $V_{ijk} = G \times_{G_{ijk}} S_{ijk}$, the tensor product $(\delta L)_{ijk} = L_{jk} L_{ik}^{-1} L_{ij}$ is the pull-back of the line bundle over G/G_{ijk}, defined by the zero weight

$$(\mu_k - \mu_j) - (\mu_k - \mu_i) + (\mu_j - \mu_i) = 0$$

of G_{ijk}. It is hence canonically trivial, with $(\delta\nabla)_{ijk}$ the trivial connection. The trivializing section $t_{ijk} = 1$ satisfies $\delta t = 1$ and $(\delta\nabla)t = 0$. Take $(B_j)_G = (\varpi_j)_G$. Then

$$(B_j)_G - (B_i)_G = (\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G = \frac{1}{2\pi i}\,\mathrm{curv}_G(\nabla^{L_{ij}})\,.$$

Thus $\mathcal{G} = (\mathcal{V}, L, t)$ is a equivariant gerbe with connection (∇, B). Since

$$\mathrm{d}_G(B_j)_G = \mathrm{d}_G(\varpi_j)_G = \eta_G|_{V_j} ,$$

this is the basic gerbe for $\mathrm{SU}(d+1)$. The transition line bundles L_{ij} may be expressed in terms of eigenspace line bundles, leading to the description of the basic gerbe from the introduction.

REMARK 5.3. This description of the basic gerbe over the special unitary group was found independently by Gawędzki-Reis [13], who also discuss the much more difficult case of quotients of $\mathrm{SU}(d+1)$ by subgroups of the center.

A similar construction works for the group $C_d = \mathrm{Sp}(d)$, the only case besides $A_d = \mathrm{SU}(d+1)$ for which the vertices of the alcove are in the weight lattice. The following table lists, for all simply connected compact simple groups, the smallest integer $k_0 > 0$ such that $k_0\mathfrak{A}$ is a weight lattice polytope[4]). The construction for $\mathrm{SU}(d+1)$ generalizes to describe the k_0-th power of the basic gerbe in all cases.

(5.2)

G	A_d	B_d	C_d	D_d	E_6	E_7	E_8	F_4	G_2
k_0	1	2	1	2	6	12	60	6	2

5.4 THE BASIC GERBE FOR GENERAL SIMPLE, SIMPLY CONNECTED G

The extra difficulty for the groups with $k_0 > 1$ comes from the fact that the pull-back maps $H^3_G(G,\mathbf{Z}) \to H^3_G(\mathcal{C}_j,\mathbf{Z}) \cong H^3_G(V_j,\mathbf{Z})$ may be a non-zero torsion class, in general. In this case the restriction of the basic gerbe to V_j will be non-trivial. Our strategy for the general case is to first construct equivariant bundle gerbes over V_j, and then glue the local data as explained in Section 4.

The centralizers G_g of elements $g \in G$ are always connected [11, Corollary (3.15)] but need not be simply-connected. The conjugacy classes $\mathcal{C}_j = q^{-1}(\mu_j)$ corresponding to the vertices of the alcove are exactly the conjugacy classes of elements for which the centralizer is semi-simple. Since

$$H^3_G(\mathcal{C}_j,\mathbf{Z}) = H^3_G(G/G_j,\mathbf{Z}) = H^3_{G_j}(\mathrm{pt},\mathbf{Z}) ,$$

we see that the torsion problem described above is related to a possibly non-trivial central extension of the centralizers G_j of $\exp(\mu_j)$ by the circle $\mathrm{U}(1)$.

---

[4]) This information is extracted from the tables in Bourbaki [5]. Letting $\varpi_1,\ldots,\varpi_d$ be the fundamental weights, one determines k_0 as the least common multiple of the numbers $\alpha_{max}\cdot\varpi_j$, using the basic inner product defined by $\alpha_{max}\cdot\alpha_{max} = 2$. The number k_0 is equal to the smallest Dynkin index of a representation $G \to \mathrm{SU}(n)$, see [28, p. 128] where the same table appears in a different context.

PROPOSITION 5.4. *Any vertex μ_j of the alcove $\mathfrak{A}$ is in the dual of the co-root lattice for the corresponding centralizer G_j. It hence defines a homomorphism $\varrho_j \in \mathrm{Hom}(\pi_1(G_j), \mathrm{U}(1))$, or equivalently a central extension of G_j by* U(1).

Proof. Let $\widetilde{G}_j$ be the universal cover of G_j. A system of simple roots for $\widetilde{G}_j$ is given by the list of all α_i $(i = 0, \dots, d)$ with $j \neq i$. The lattice Λ_j is spanned by the corresponding coroots $\check{\alpha}_i$. To show that μ_j is in the dual of the co-root lattice, we have to verify that $\langle \mu_j, \check{\alpha}_i \rangle \in \mathbf{Z}$ for $i \neq j$. For $i \neq 0, j$ this is obvious since $\mu_j(\check{\alpha}_i) = 0$. For $i = 0$, we have $\|\check{\alpha}_0\|^2 = 2$, and therefore $\check{\alpha}_0 = \alpha_0$ and $\mu_j(\check{\alpha}_0) = \alpha_0(\mu_j) = -1$.

Recall that for $i \neq j$, G_{ij} is the centralizer of points $\exp \mu$ with $\mu = t\mu_j + (1-t)\mu_i$ for some $0 < t < 1$. Let $\varrho_{ij} \in \mathrm{Hom}(\pi_1(G_{ij}), \mathrm{U}(1))$ be the quotient of $\pi_1(G_{ij}) \to \pi_1(G_j) \xrightarrow{\varrho_j} \mathrm{U}(1)$ by the homomorphism $\pi_1(G_{ij}) \to \pi_1(G_i) \xrightarrow{\varrho_i} \mathrm{U}(1)$.

LEMMA 5.5. *The difference $\mu_j - \mu_i \in \mathfrak{g}_{ij}$ is fixed under G_{ij}, and $\varrho_{ij} \in \mathrm{Hom}(\pi_1(G_{ij}), \mathrm{U}(1))$ is its image under the exact sequence* (3.2) *for $K = G_{ij}$.*

Proof. Since G_{ij} fixes the curve $g(t) = \exp(t\mu_j + (1-t)\mu_i) = \exp(\mu_i)\exp(t(\mu_j - \mu_i))$, it stabilizes the Lie algebra element $\mu_j - \mu_i$. The second claim is immediate from the definition.

We are now in position to explain our construction of the basic gerbe in the general case. For all $I \subset \{0, \dots, d\}$ let $X_I \to V_I$ be the G-equivariant principal G_I-bundle,

$$X_I = G \times S_I \to V_I = G \times_{G_I} S_I .$$

X_I is the pull-back of the G_I-bundle $G \to G/G_I$, and in particular carries a G-invariant connection θ_I obtained by pull-back of the unique G-invariant connection on that bundle. For $I \supset J$ there are natural G-equivariant inclusions $f_I^J \colon X_I \to X_J$, and these are compatible as in Section 4. The homomorphisms $\varrho_j \colon \pi_1(G_j) \to \mathrm{U}(1)$ define flat, G-equivariant bundle gerbes $\mathcal{G}_j = (X_j, L_j, t_j)$ over V_j.

The quotient of the two gerbes on V_{ij}, obtained by pulling back $\mathcal{G}_i$, $\mathcal{G}_j$ to X_{ij}, is just the gerbe defined by the homomorphism $\varrho_{ij} \colon \pi_1(G_{ij}) \to \mathrm{U}(1)$. By Lemma 5.5 and Proposition 3.2(b), it follows that this quotient gerbe has a distinguished, equivariant pseudo-line bundle (E_{ij}, s_{ij}) (where E_{ij} is trivial), with connection $\nabla^{E_{ij}}$ induced from the connection θ_{ij}. From the definition of θ_{ij}, it follows that the equivariant error 2-form for this connection is the pull-back of the equivariant symplectic form on the coadjoint orbit through $\mu_j - \mu_i$.

We now modify the bundle gerbe connection by adding the equivariant 2-form $(\varpi_j)_G \in \Omega^2_G(V_j)$ to the gerbe connection. Proposition 5.2(d) shows that the equivariant error 2-form of $\nabla^{E_{ij}}$ with respect to the new gerbe connection vanishes. The other conditions from the gluing construction in §4 are trivially satisfied. Since the equivariant 3-curvature for the new gerbe connection on $\mathcal{G}_j$ is $d_G(\varpi_j)_G = \eta_G|_{V_j}$, we have constructed an equivariant bundle gerbe with connection, with equivariant curvature-form η_G.

REMARK 5.6. For $G = \mathrm{SU}(d+1)$ this construction reduces to the construction in terms of transition line bundles: All L_i, t_i, E_{ij}, u_{ijk} are trivial in this case, hence the entire information on the gerbe resides in the functions $s_{ij}\colon (X_{ij})^{[2]} \to \mathrm{U}(1)$ defined by the differences $\mu_j - \mu_i$. The condition $\delta s_{ij} = 1$ for these functions means that s_{ij} defines a line bundle L_{ij} over V_{ij}, as remarked at the beginning of Section 2.2. The condition $s_{ij}s_{jk}s_{ki} = 1$ over X_{ijk} is the compatibility condition over triple intersections.

6. PRE-QUANTIZATION OF CONJUGACY CLASSES

It is a well-known fact from symplectic geometry that a coadjoint orbit $\mathcal{O} = G.\mu$ through $\mu \in \mathfrak{t}^*_+$ has integral symplectic form, i.e. admits a pre-quantum line bundle, if and only if μ is in the weight lattice Λ^*. The analogous question for conjugacy classes reads: For which $\mu \in \mathfrak{A}$ and $m \in \mathbf{N}$ does the pull-back of the mth power of the basic gerbe $\mathcal{G}^m$ to the conjugacy class $\mathcal{C} = G.\exp(\mu)$ admit a pseudo-line bundle, with $m\omega_{\mathcal{C}}$ as its error 2-form? For any positive integer $m > 0$ let

$$\Lambda^*_m = \Lambda^* \cap m\mathfrak{A}$$

be the set of level m weights. As is well-known [26], the set Λ^*_m parametrizes the positive energy representations of the loop group LG at level m.

THEOREM 6.1. *The restriction of $\mathcal{G}^m$ to a conjugacy class $\mathcal{C}$ admits a pseudo-line bundle $\mathcal{L}$ with connection, with error 2-form $m\omega_{\mathcal{C}}$, if and only if $\mathcal{C} = G.\exp(\mu/m)$ with $\mu \in \Lambda^*_m$. Moreover $\mathcal{L}$ has an equivariant extension in this case, with $m\omega_{\mathcal{C}}$ as its* equivariant *error 2-form* .