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Autor: Meinrenken, Eckhard
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5.2 THE BASIC 3-FORM ON G

Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left- and right-invariant Maurer-Cartan forms on G , respectively. The 3-form $\eta \in \Omega^3(G)$ given by³⁾

$$\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L] = \frac{1}{12} \theta^R \cdot [\theta^R, \theta^R]$$

is closed, and has a closed equivariant extension $\eta_G \in \Omega_G^3(G)$ given by

$$\eta_G(\xi) := \eta - \frac{1}{2}(\theta^L + \theta^R) \cdot \xi.$$

Their cohomology classes represent generators of $H^3(G, \mathbf{Z}) = \mathbf{Z}$ and $H_G^3(G, \mathbf{Z}) = \mathbf{Z}$, respectively. The pull-back of η_G to any conjugacy class $\iota_C: C \hookrightarrow G$ is exact. In fact, let $\omega_C \in \Omega^2(C)^G \subset \Omega_G^2(C)$ be the invariant 2-form given on generating vector fields ξ_C, ξ'_C for $\xi, \xi' \in \mathfrak{g}$ by the formula

$$\omega_C(\xi_C(g), \xi'_C(g)) = \frac{1}{2} \xi \cdot (\text{Ad}_g - \text{Ad}_{g^{-1}}) \xi'.$$

Then [1, 16]

$$d_G \omega_C + \iota_C^* \eta_G = 0.$$

We will now show that η_G is exact over each of the open subsets V_j . Let $C_j = q^{-1}(\mu_j) \subset V_j$ be the conjugacy classes corresponding to the vertices.

LEMMA 5.1. *The linear retraction*

$$[0, 1] \times \mathfrak{A}_j \rightarrow \mathfrak{A}_j, \quad (t, \mu_j + \zeta) \mapsto \mu_j + (1 - t) \zeta$$

of $\mathfrak{A}_j$ onto the vertex μ_j lifts uniquely to a smooth G -equivariant retraction from V_j onto C_j .

Proof. Recall that the slice S_j is an open neighborhood of $\exp(\mu_j)$ in G_j . Any G_j -equivariant retraction from S_j onto $\exp \mu_j$ extends uniquely to a G -equivariant retraction from $V_j = G \times_{G_j} S_j$ onto C_j . Note that $S'_j = G_j \cdot (\mathfrak{A}_j - \mu_j)$ is a star-shaped open neighborhood of 0 in $\mathfrak{g}_j$, and that $S'_j \rightarrow S_j, \zeta \mapsto \exp(\mu_j) \exp(\zeta)$ is a G_j -equivariant diffeomorphism. The linear retraction of S'_j onto the origin gives the desired retraction of S_j . Uniqueness is clear, since the retraction has to preserve $\exp(\mathfrak{A}_j) \subset V_j$, by equivariance.

³⁾ For $\mathfrak{g}$ -valued forms β_1, β_2 , the bracket $[\beta_1, \beta_2]$ denotes the $\mathfrak{g}$ -valued form obtained by applying the Lie bracket $[\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ to the $\mathfrak{g} \otimes \mathfrak{g}$ -valued form $\beta_1 \wedge \beta_2$.

Let

$$\mathbf{h}_j: \Omega^p(V_j) \rightarrow \Omega^p([0, 1] \times V_j) \rightarrow \Omega^{p-1}(V_j)$$

be the de Rham homotopy operator for this retraction, given (up to a sign) by pull-back under the retraction, followed by integration over the fibers of $[0, 1] \times V_j \rightarrow V_j$. It has the property

$$(5.1) \quad d_G \mathbf{h}_j + \mathbf{h}_j d_G = \text{Id} - \pi_j^* \iota_j^*$$

where $\iota_j: C_j \rightarrow V_j$ is the inclusion and $\pi_j: V_j = G \times_{G_j} S_j \rightarrow G/G_j = C_j$ the projection. Let $(\varpi_j)_G = \mathbf{h}_j \eta_G - \pi_j^* \omega_{C_j} \in \Omega_G^2(V_j)$, and write $(\varpi_j)_G = \varpi_j - \Psi_j$ where $\varpi_j \in \Omega^2(V_j)$ and $\Psi_j \in \Omega^0(V_j, \mathfrak{g})$.

PROPOSITION 5.2. *The equivariant 2-form $(\varpi_j)_G = \varpi_j - \Psi_j$ has the following properties.*

$$(a) \quad d_G(\varpi_j)_G = \eta_G.$$

(b) *The pull-back of $(\varpi_j)_G$ to a conjugacy class $C \subset V_j$ is given by*

$$\iota_C^*(\varpi_j)_G = \Psi_j^*(\omega_{\mathcal{O}})_G - \omega_C,$$

where $(\omega_{\mathcal{O}})_G$ is the equivariant symplectic form on the adjoint orbit $\mathcal{O} = \Psi_j(C)$,

(c) *The pull-back of Ψ_j to the conjugacy class C_j vanishes. In fact, $\Psi_j(\exp \xi) = \xi - \mu_j$ for all $\xi \in \mathfrak{A}_j$.*

(d) *Over each intersection $V_{ij} = V_i \cap V_j$, the difference $\Psi_i - \Psi_j$ takes values in the adjoint orbit $\mathcal{O}_{ij}$ through $\mu_j - \mu_i \in \mathfrak{g} \cong \mathfrak{g}^*$. Furthermore,*

$$(\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G$$

where $p_{ij}: V_{ij} \rightarrow \mathcal{O}_{ij}$ is the map defined by $\Psi_i - \Psi_j$, and $(\omega_{\mathcal{O}_{ij}})_G$ is the equivariant symplectic form on the orbit.

Proof. (a) holds by construction. (b) follows from the observation that $\iota_C^*(\varpi_j)_G + \omega_C$ is an equivariantly closed 2-form on C_j , with Ψ_j as its moment map. To prove (c) we note that since the retraction is equivariant, we have $\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* = (\exp|_{\mathfrak{A}_j})^* \circ \mathbf{h}_j$ where $(\exp|_{\mathfrak{A}_j})^*$ is pull-back to $\mathfrak{A}_j \subset \mathfrak{t}$ and where $\tilde{\mathbf{h}}_j$ is the homotopy operator for the linear retraction of $\mathfrak{t}$ onto $\{\mu_j\}$. Let $\nu: \mathfrak{A}_j \rightarrow \mathfrak{t}$ be the coordinate function (inclusion). Then

$$\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* \frac{1}{2}(\theta^L + \theta^R) = \tilde{\mathbf{h}}_j \circ d\nu = \nu - \mu_j,$$

proving that $(\exp|_{\mathfrak{A}_j})^* \Psi_j = \nu - \mu_j$. This yields (c), by equivariance. For $\nu \in \mathfrak{A}_{ij}$ we have, using (c),

$$(\Psi_i - \Psi_j)(\exp \nu) = (\nu - \mu_i) - (\nu - \mu_j) = \mu_j - \mu_i.$$

By equivariance, it follows that $\Psi_i - \Psi_j$ takes values in the adjoint orbit through $\mu_j - \mu_i$. The difference $\varpi_i - \varpi_j$ vanishes on the maximal torus T , and is therefore determined by its contractions with generating vector fields. Since $\Psi_i - \Psi_j$ is a moment map for $\varpi_i - \varpi_j$, it follows that $\varpi_i - \varpi_j$ equals the pull-back of the symplectic form on $G \cdot (\mu_j - \mu_i)$.

5.3 THE SPECIAL UNITARY GROUP

For the special unitary group $G = \mathrm{SU}(d+1)$, the construction of the basic gerbe simplifies due to the fact that in this case all vertices μ_j of the alcove are contained in the weight lattice. In fact the gerbe is presented as a Chatterjee-Hitchin gerbe for the cover $\mathcal{V} = \{V_i, i = 0, \dots, d\}$.

For each weight $\mu \in \Lambda^* \subset \mathfrak{t} \subset \mathfrak{g}$, let G_μ be its stabilizer for the adjoint action and let $\mathbf{C}_\mu$ the 1-dimensional G_μ -representation with infinitesimal character μ . Let the line bundle $L_\mu = G \times_{G_\mu} \mathbf{C}_\mu$ equipped with the unique left-invariant connection ∇ . Then L_μ is a G -equivariant pre-quantum line bundle for the orbit $\mathcal{O} = G \cdot \mu$. That is,

$$\frac{i}{2\pi} \mathrm{curv}_G(\nabla) = (\omega_{\mathcal{O}})_G := \omega_{\mathcal{O}} - \Phi_{\mathcal{O}}$$

where $\omega_{\mathcal{O}}$ is the symplectic form and $\Phi_{\mathcal{O}}: \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is the moment map given as inclusion.

In particular, in the case of $\mathrm{SU}(d+1)$ all orbits $\mathcal{O}_{ij} = G \cdot (\mu_j - \mu_i)$ carry G -equivariant pre-quantum line bundles. Recall the fibrations $p_{ij}: V_{ij} \rightarrow \mathcal{O}_{ij}$ defined by $\Psi_i - \Psi_j$, and let

$$L_{ij} = p_{ij}^*(L_{\mu_j - \mu_i}),$$

equipped with the pull-back connection. For any triple intersection $V_{ijk} = G \times_{G_{ijk}} S_{ijk}$, the tensor product $(\delta L)_{ijk} = L_{jk} L_{ik}^{-1} L_{ij}$ is the pull-back of the line bundle over G/G_{ijk} , defined by the zero weight

$$(\mu_k - \mu_j) - (\mu_k - \mu_i) + (\mu_j - \mu_i) = 0$$

of G_{ijk} . It is hence canonically trivial, with $(\delta \nabla)_{ijk}$ the trivial connection. The trivializing section $t_{ijk} = 1$ satisfies $\delta t = 1$ and $(\delta \nabla)t = 0$. Take $(B_j)_G = (\varpi_j)_G$. Then

$$(B_j)_G - (B_i)_G = (\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G = \frac{1}{2\pi i} \mathrm{curv}_G(\nabla^{L_{ij}}).$$

Thus $\mathcal{G} = (\mathcal{V}, L, t)$ is a equivariant gerbe with connection (∇, B) . Since