3.1 The cuts

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The proof of Theorem 2 now proceeds in two stages. First, (in 3.1) we look for all possible places where we could cut a rational knot K open to a rational tangle, and we show that all cuts that open K to other rational tangles give tangles arithmetically equivalent to the tangle T. Second, (in 3.2) given two isotopic reduced alternating rational knot diagrams, we have to check that the rational tangles that they open to are arithmetically equivalent. By the solution to the Tait Conjecture these isotopic knot diagrams will differ by a sequence of flypes. So we analyze what happens when a flype is performed on K.

3.1 THE CUTS

Let K be a rational knot that is the numerator closure of a rational tangle T. We will look for all 'rational' cuts on K. In our study of cuts we shall assume that T is in reduced canonical form. The more general case where T is in reduced alternating twist form is completely analogous and we make a remark at the end of the subsection. Moreover, the cut analysis in the case where $a_1 = 0$ is also completely analogous for all cuts with appropriate adjustments. There are three types of rational cuts.

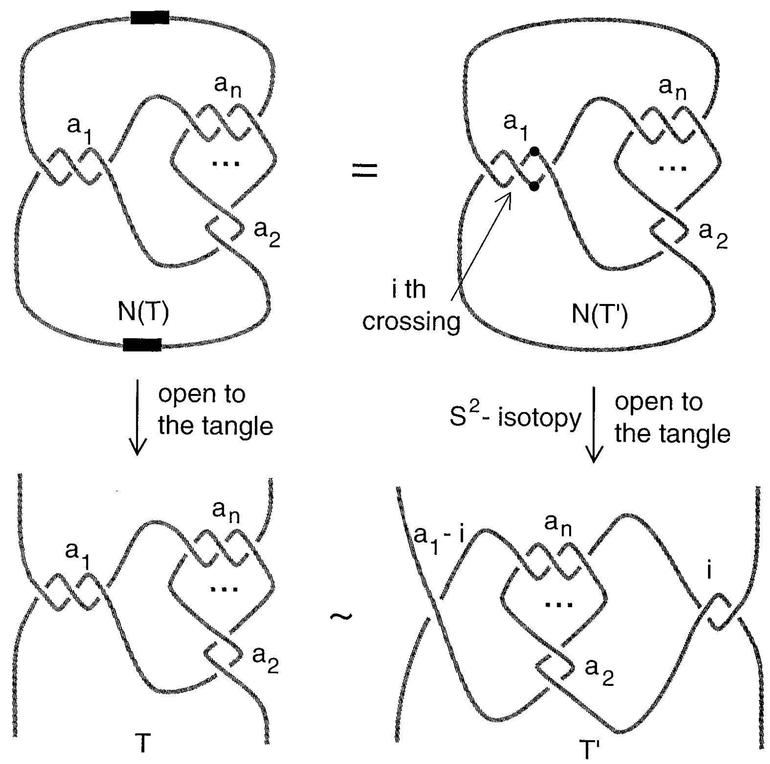

FIGURE 14
Standard cuts

THE STANDARD CUTS. The tangle $T = [[a_1], [a_2], \ldots, [a_n]]$ is said to arise as *the standard cut* on $K = N(T)$. If we cut K at another pair of 'vertical' points that are adjacent to the ith crossing of the elementary tangle $[a_1]$ (counting from the outside towards the inside of T) we obtain the alternating rational tangle in twist form $T' = [[a_1 - i], [a_2], \ldots, [a_n]] + [i]$. Clearly, this tangle is isotopic to T by a sequence of flypes that send all the horizontal twists to the right of the tangle. See the right hand illustration of Figure 14 for $i = 2$. Thus, by the Conway Theorem, T' will have the same fraction as T. Any such cut on K shall be called a *standard cut* on K.

THE SPECIAL CUTS. A key example of the arithmetic relationship of the classification of rational knots is illustrated in Figure 15. The two tangles $T = [-3]$ and $S = [1] + \frac{1}{[2]}$ are non-isotopic by the Conway Theorem, since $F(T) = -3 = 3/-1$, while $F(S) = 1 + 1/2 = 3/2$. But they have isotopic numerators: $N(T) \sim N(S)$, the left-handed trefoil. Now $-1 \equiv 2 \mod 3$, confirming Theorem 2.

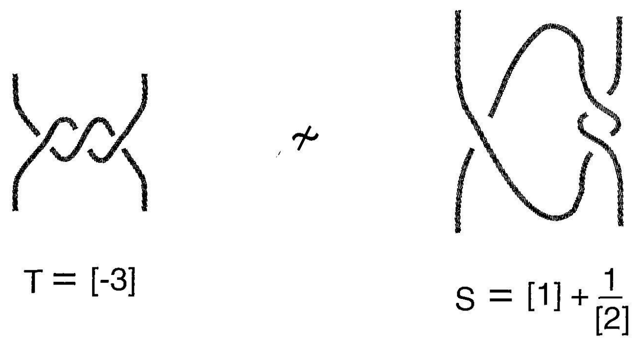

FIGURE 15
An example of the special cut

We now analyze the above example in general. Let $K = N(T)$, where $T = [[a_1], [a_2], \ldots, [a_n]]$. Since T is assumed to be in reduced form, it follows that $a_1 \neq 0$, so T can be written in the form $T = [+1] + R$ or $T = [-1] + R$, and the tangle R is also rational.

The indicated horizontal crossing $[+1]$ of the tangle $T = [+1] + R$, which is the first crossing of $[a_1]$ and the last created crossing of T, may also be seen as a vertical one. So, instead of cutting the diagram K open at the two standard cutpoints to obtain the tangle T, we cut at the two other marked 'horizontal' points on the first crossing of the subtangle $[a_1]$ to obtain a new 2-tangle T' (see Figure 16). T' is clearly rational, since R is rational. The tangle T' is said to arise as *the special cut* on K.

We would like to identify this rational tangle T'. For this reason we first swing the upper arc of K down to the bottom of the diagram in order to free the region of the cutpoints. By our convention for the signs of crossings in

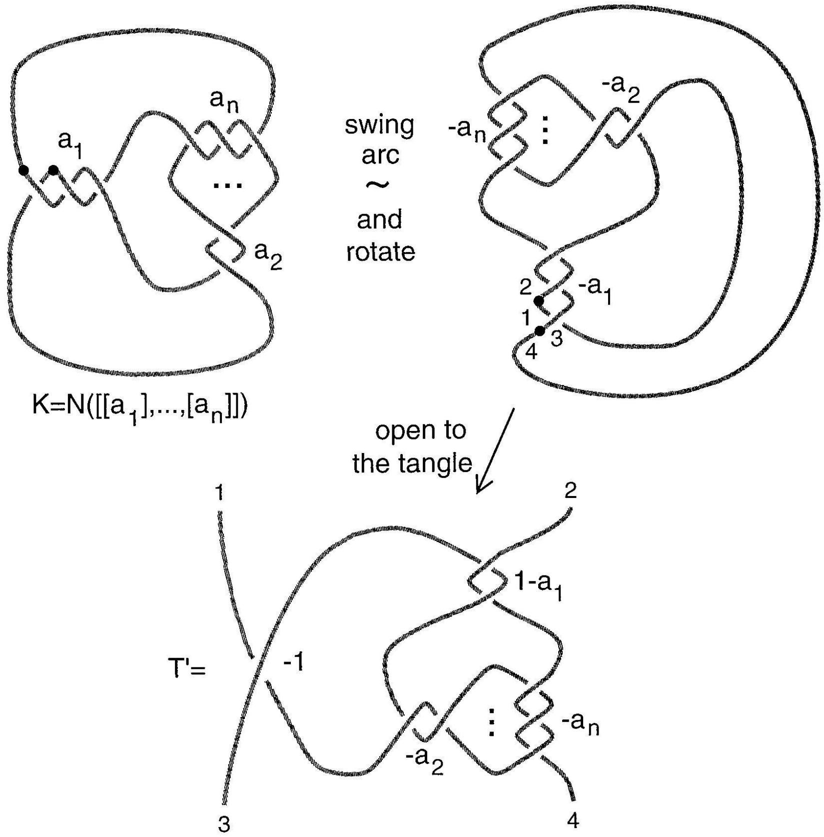

FIGURE 16
Preparing for the special cut

terms of the checkerboard shading, this forces all crossings of T to change sign from positive to negative and vice versa. We then rotate K by $90°$ on its plane (see right-hand illustration of Figure 16). This forces all crossings of T to change from horizontal to vertical and vice versa. In particular, the marked crossing $[+1]$, that was seen as a vertical one in T, will now look as a horizontal $[-1]$ in T'. In fact, this will be the only last horizontal crossing of T', since all other crossings of $[a_1]$ are now vertical. So, if $T = [[a_1], [a_2], \ldots, [a_n]]$ then $R = [[a_1 - 1], [a_2], \ldots, [a_n]]$ and

$$T' = [[-1], [1 - a_1], [-a_2], \ldots, [-a_n]] .$$

Note that if the crossings of K were all of negative type, thus all the a_i's would be negative, the tangle T' would be $T' = [[+1], [-1 - a_1], [-a_2], \ldots, [-a_n]]$. In the example of Figure 15 if we took $R = [-2]$, then $T = [-1] + R$ and $T' = S = [[+1], [+2]]$.

The special cut is best illustrated in Figure 17. We consider the rational knot diagram $K = N([+1] + R)$. (We analyze $N([-1] + R)$ in the same way.) As we see here, a sequence of isotopies and cutting K open allow us to read the new tangle:

$$T' = [-1] - \frac{1}{R} .$$

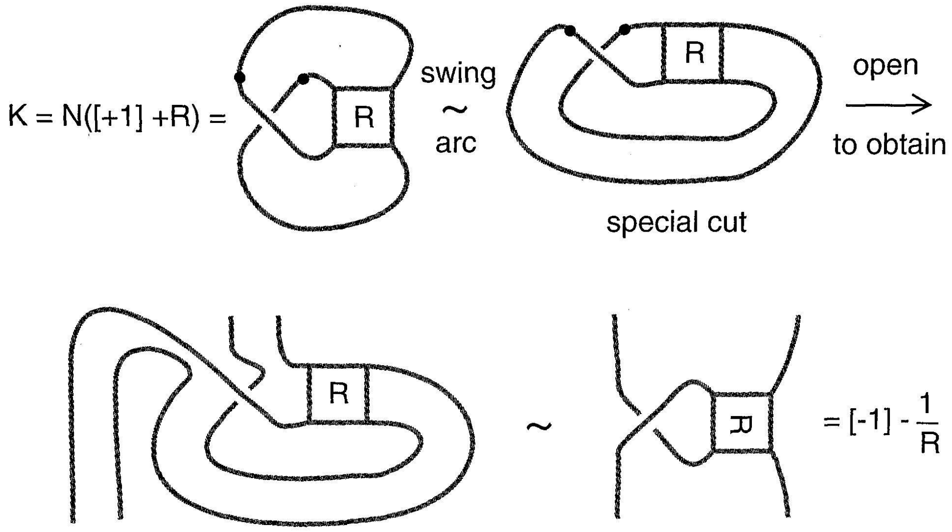

FIGURE 17
The tangle of the special cut

From the above we have $N([+1]+R) \sim N([-1]-\frac{1}{R})$. Let now the fractions of T, R and T' be $F(T) = P/Q$, $F(R) = p/q$ and $F(T') = P'/Q'$ respectively. Then

$$F(T) = F([+1]+R) = 1 + p/q = (p+q)/q = P/Q,$$

while

$$F(T') = F([-1] - 1/R) = -1 - q/p = (p+q)/(-p) = P'/Q'.$$

The two fractions are different, thus the two rational tangles that give rise to the same rational knot are not isotopic. We observe that $P = P'$ and

$$q \equiv -p \mod (p+q) \iff Q \equiv Q' \mod P.$$

This arithmetic equivalence demonstrates another case for Theorem 2. Notice that, although both the bottom twist and the special cut fall into the same arithmetic equivalence, the arithmetic of the special cut is more subtle than the arithmetic of the bottom twist.

If we cut K at the two lower horizontal points of the first crossing of $[a_1]$ we obtain the same rational tangle T'. Also, if we cut at any other pair of upper or lower horizontal adjacent points of the subtangle $[a_1]$ we obtain a rational tangle in twist form isotopic to T'. Such a cut shall be called a *special cut*. See Figure 18 for an example. Finally, we may cut K at any pair of upper or lower horizontal adjacent points of the subtangle $[a_n]$. We shall call this a *special palindrome cut*. We will discuss this case after having analyzed the last type of a cut, the palindrome cut.

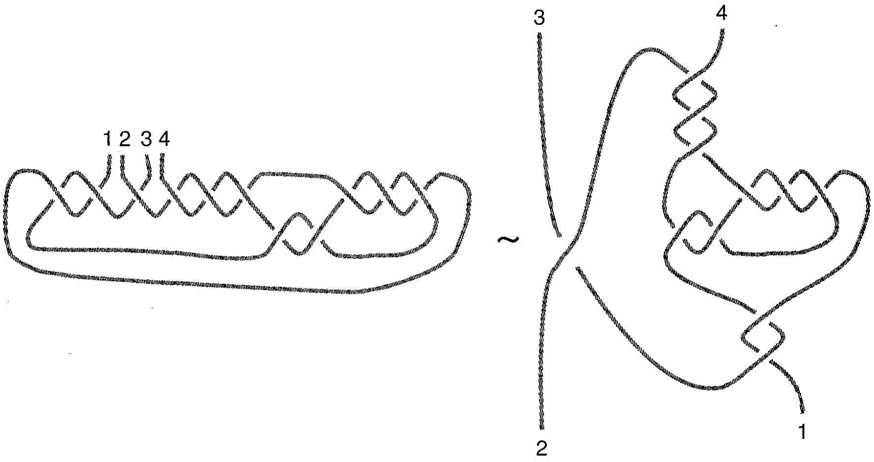

FIGURE 18
A special cut

NOTE. We would like to point out that the horizontal-vertical ambiguity of the last crossing of a rational tangle $T = [[a_1], \ldots, [a_{n-1}], [a_n]]$, which with the special cut on $K = N(T)$ gives rise to the tangle $[[\mp 1], [\pm 1 - a_1], [-a_2], \ldots, [-a_n]]$, is very similar to the horizontal-vertical ambiguity of the first crossing that does not change the tangle and it gives rise to the tangle continued fraction $[[a_1], \ldots, [a_{n-1}], [a_n \mp 1], [\pm 1]]$.

REMARK 2. A special cut is nothing more than the addition of a bottom twist. Indeed, as Figure 19 illustrates, applying a positive bottom twist to the tangle T' of the special cut yields the tangle $S = ([-1] - 1/R) * [+1]$, and we find that if $F(R) = p/q$ then $F([+1] + R) = (p + q)/q$ while $F(([-1] - 1/R) * [+1]) = 1/(1 + 1/(-1 - q/p)) = (p + q)/q$. Thus we see that the fractions of $T = [+1] + R$ and $S = ([-1] - 1/R) * [+1]$ are equal and by the Conway Theorem the tangle S is isotopic to the original tangle T of the standard cut. The isotopy move is nothing but the transfer move of Figure 11. The isotopy is illustrated in Figure 19. Here we used the Flipping Lemma.

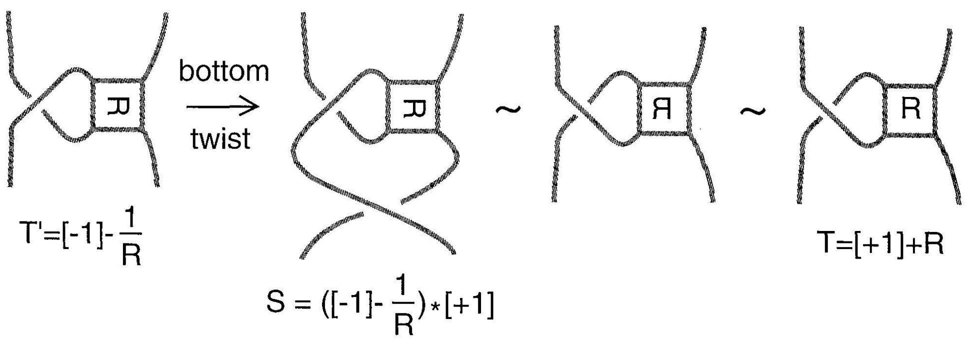

FIGURE 19
Special cuts and bottom twists

THE PALINDROME CUTS. In Figure 20 we see that the tangles

$$T = [[2],[3],[4]] = [2] + \frac{1}{[3] + \frac{1}{[4]}}$$

and

$$S = [[4],[3],[2]] = [4] + \frac{1}{[3] + \frac{1}{[2]}}$$

both have the same numerator closure. This is another key example of the basic relationship given in the classification of rational knots.

In the general case if $T = [[a_1],[a_2],\ldots,[a_n]]$, we shall call the tangle $S = [[a_n],[a_{n-1}],\ldots,[a_1]]$ *the palindrome of* T. Clearly these tangles have the same numerator: $K = N(T) = N(S)$. Cutting open K to yield T is the standard cut, while cutting to yield S shall be called *the palindrome cut* on K.

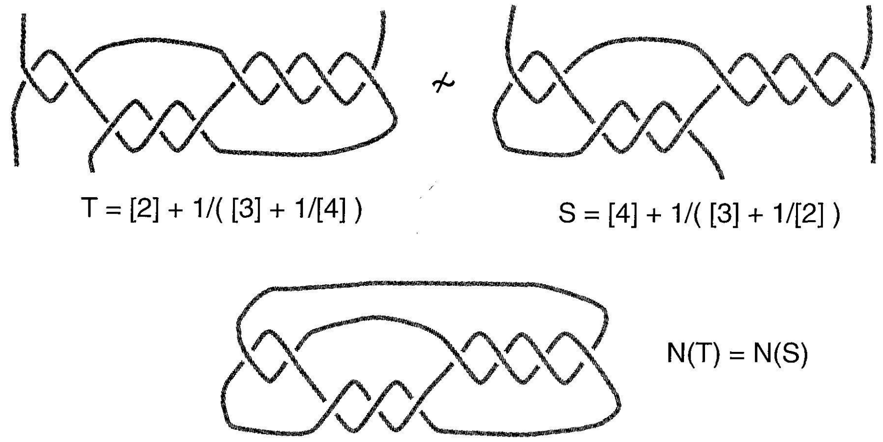

FIGURE 20
An instance of the palindrome equivalence

The tangles in Figure 20 have corresponding fractions

$$F(T) = 2 + \frac{1}{3 + \frac{1}{4}} = \frac{30}{13} \quad \text{and} \quad F(S) = 4 + \frac{1}{3 + \frac{1}{2}} = \frac{30}{7}.$$

Note that $7 \cdot 13 \equiv 1 \mod 30$. This is the other instance of the arithmetic behind the classification of rational knots in Theorem 2. In order to check the arithmetic in the general case of the palindrome cut we need to generalize this pattern to arbitrary continued fractions and their palindromes (obtained by reversing the order of the terms). Then we have the following

THEOREM 4 (Palindrome Theorem). *Let* $\{a_1, a_2, \ldots, a_n\}$ *be a collection of* n *non-zero integers, and let* $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ *and* $\frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1]$. *Then* $P = P'$ *and* $QQ' \equiv (-1)^{n+1} \mod P$.

The Palindrome Theorem is a known result about continued fractions. For example see [35] or [16], p. 25, Exercise 2.1.9. We shall give here our proof of this statement. For this we will first present a way of evaluating continued fractions via 2×2 matrices (compare with [11], [18]). This method of evaluation is crucially important in our work in the rest of the paper. Let $\frac{p}{q} = [a_2, a_3, \ldots, a_n]$. Then we have:

$$[a_1, a_2, \ldots, a_n] = a_1 + \frac{1}{\frac{p}{q}} = a_1 + \frac{q}{p} = \frac{a_1 p + q}{p} = \frac{p'}{q'}.$$

Taking the convention that $\left[\binom{p}{q}\right] := \frac{p}{q}$, with our usual conventions for formal fractions such as $\frac{1}{0}$, we can thus write a corresponding matrix equation in the form

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a_1 p + q \\ p \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

We let

$$M(a_i) = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrices $M(a_i)$ are said to be the *generating matrices* for continued fractions, as we have:

LEMMA 1 (Matrix interpretation for continued fractions). *For any sequence of non-zero integers $\{a_1, a_2, \ldots, a_n\}$ the value of the corresponding continued fraction is given through the following matrix product*

$$[a_1, a_2, \ldots, a_n] = [M(a_1)M(a_2)\cdots M(a_n)\cdot v]$$

where

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proof. We observe that

$$\left[M(a_n)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} a_n \\ 1 \end{pmatrix}\right] = a_n = [a_n]$$

and

$$\left[M(a_{n-1})\begin{pmatrix} a_n \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} a_{n-1}a_n + 1 \\ a_n \end{pmatrix}\right] = [a_{n-1}, a_n].$$

Now the lemma follows by induction. $\square$

Proof of the palindrome theorem. We wish to compare $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ and $\frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1]$. By Lemma 1 we can write

$$\frac{P}{Q} = [M(a_1)M(a_2)\cdots M(a_n) \cdot v] \quad \text{and} \quad \frac{P'}{Q'} = [M(a_n)M(a_{n-1})\cdots M(a_1) \cdot v] .$$

Let

$$M = M(a_1)M(a_2)\cdots M(a_n)$$

and

$$M' = M(a_n)M(a_{n-1})\cdots M(a_1) .$$

Then $\frac{P}{Q} = [M \cdot v]$ and $\frac{P'}{Q'} = [M' \cdot v]$. We observe that

$$\begin{aligned} M^T &= (M(a_1)M(a_2)\cdots M(a_n))^T = (M(a_n))^T(M(a_{n-1}))^T \cdots (M(a_1))^T \\ &= M(a_n)M(a_{n-1})\cdots M(a_1) = M' , \end{aligned}$$

since $M(a_i)$ is symmetric, where M^T is the transpose of T. Thus

$$M' = M^T .$$

Let

$$M = \begin{pmatrix} X & Y \\ Z & U \end{pmatrix} .$$

In order that the equations $[M \cdot v] = \frac{P}{Q}$ and $[M^T \cdot v] = \frac{P'}{Q'}$ are satisfied it is necessary that $X = P$, $X = P'$, $Z = Q$ and $Y = Q'$. That is, we should have:

$$M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} P & Q \\ Q' & U \end{pmatrix} .$$

Furthermore, since the determinant of $M(a_i)$ is equal to -1, we have that

$$\det(M) = (-1)^n .$$

Thus

$$PU - QQ' = (-1)^n ,$$

so that

$$QQ' \equiv (-1)^{n+1} \mod P ,$$

and the proof of the Theorem is complete. □

REMARK 3. Note in the argument above that the entries of the matrix $M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}$ of a given continued fraction $[a_1, a_2, \ldots, a_n] = \frac{P}{Q}$ involve also the evaluation of its palindrome $[a_n, a_{n-1}, \ldots, a_1] = \frac{P}{Q'}$.

Returning now to the analysis of the palindrome cut, we apply Theorem 4 in order to evaluate the fraction of palindrome rational tangles $T = [\frac{P}{Q}]$ and $S = [\frac{P'}{Q'}]$. From the above analysis we have $P = P'$. Also, by our assumption these tangles have continued fraction forms with odd length n, so we have the relation

$$QQ' \equiv 1 \mod P$$

and this is the second of the arithmetic relations of Theorem 2.

If we cut $K = N(T)$ at any other pair of 'vertical' points of the subtangle $[a_n]$ we obtain a rational tangle in twist form isotopic to the palindrome tangle S. Any such cut shall be called a *palindrome cut.*

Having analyzed the arithmetic of the palindrome cuts we can now return to the special palindrome cuts on the subtangle $[a_n]$. These may be considered as special cuts on the palindrome tangle S. So, the fraction of the tangle of such a cut will satisfy the first type of arithmetic relation of Theorem 2 with the fraction of S, namely a relation of the type $q \equiv q' \mod p$, which, consequently, satisfies the second type of arithmetic relation with the fraction of T, namely a relation of the type $qq' \equiv 1 \mod p$. In the end a special palindrome cut will satisfy an arithmetic relation of the second type. This concludes the arithmetic study of the rational cuts.

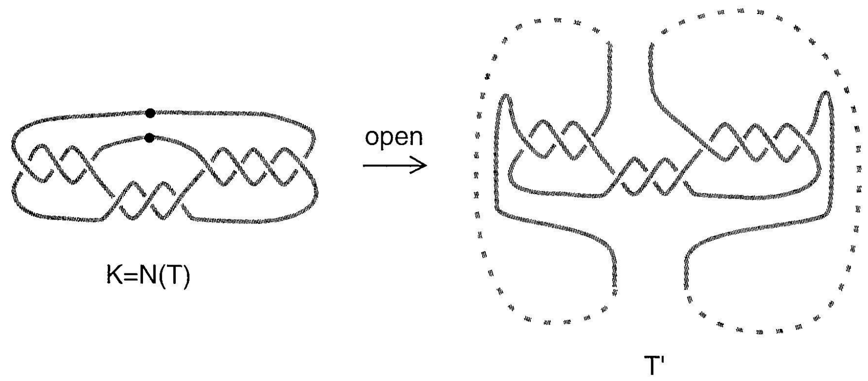

FIGURE 21

A non-rational cut

We now claim that the above listing of the three types of rational cuts is a complete catalog of cuts that can open the link K to a rational tangle: the standard cuts, the special cuts and the palindrome cuts. This is the crux of our proof.

In Figure 21 we illustrate one example of a cut that is not rational. This is a possible cut made in the middle of the representative diagram $N(T)$. Here we see that if T' is the tangle obtained from this cut, so that $N(T') = K$, then $D(T')$ is a connected sum of two non-trivial knots. Hence the denominator $K' = D(T')$ is not prime. Since we know that both the numerator and the denominator of a rational tangle are prime (see [5], p. 91 or [19], Chapter 4, pp. 32–40), it follows that T' is not a rational tangle. Clearly the above argument is generic. It is not hard to see by enumeration that all possible cuts with the exception of the ones we have described will not give rise to rational tangles. We omit the enumeration of these cases.

This completes the proof that all of the rational tangles that close to a given standard rational knot diagram are arithmetically equivalent.

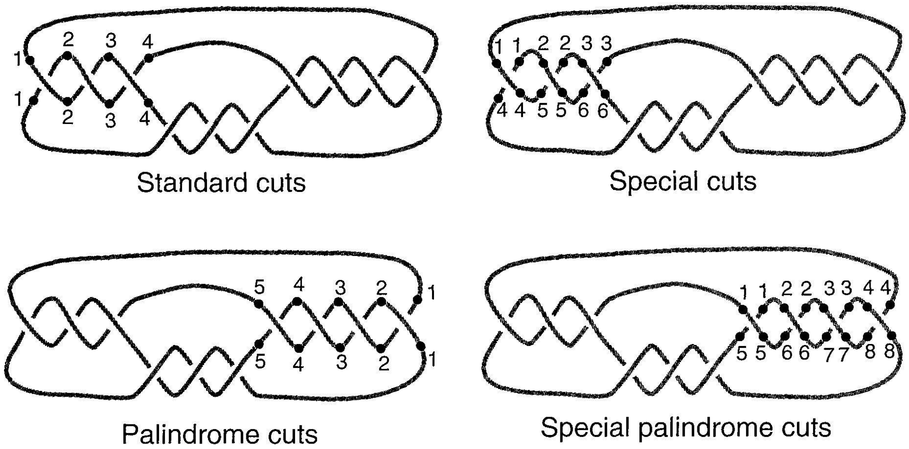

FIGURE 22
Standard, special, palindrome and special palindrome cuts

In Figure 22 we illustrate on a representative rational knot in 3-strand-braid form all the cuts that exhibit that knot as a closure of a rational tangle. Each pair of points is marked with the same number. •

REMARK 4. It follows from the above analysis that if T is a rational tangle in twist form, which is isotopic to the standard form $[[a_1], [a_2], \ldots, [a_n]]$, then all arithmetically equivalent rational tangles can arise by any cut of the above types either on the crossings that add up to the subtangle $[a_1]$ or on the crossings of the subtangle $[a_n]$.

3.2 THE FLYPES

Diagrams for knots and links are represented on the surface of the two-sphere, S^2, and then notationally on a plane for purposes of illustration.