1.1 Définition of quasi-invariants

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proofs are postponed until Lecture 3). In Lecture 2, we explain the origin of the ring of quasi-invariants in the theory of integrable systems, and introduce some tools from integrable systems, such as the Baker-Akhieser function. Finally, in Lecture 3, we develop the theory of the rational Cherednik algebra, the representation-theoretic techniques due to Opdam and Rouquier, and finish the proofs of the geometric statements from Chapter 1.

1. LECTURE 1

1.1 DEFINITION OF QUASI-INVARIANTS

In this lecture we will define the ring of quasi-invariants Q_m and discuss its main properties.

We will work over the field $\mathbf{C}$ of complex numbers. Let W be a finite Coxeter group, i.e. a finite group generated by reflections. Let us denote by $\mathfrak{h}$ its reflection representation. A typical example is the Weyl group of a semisimple Lie algebra acting on a Cartan subalgebra $\mathfrak{h}$. In the case the Lie algebra is $\mathfrak{sl}(n)$, we have that W is the symmetric group S_n on n letters and $\mathfrak{h}$ is the space of diagonal traceless $n \times n$ matrices.

Let $\Sigma \subset W$ denote the set of reflections. Clearly, W acts on Σ by conjugation. Let $m \colon \Sigma \to \mathbf{Z}_+$ be a function on Σ taking non negative integer values, which is W-invariant. The number of orbits of W on Σ is generally very small. For example, if W is the Weyl group of a simple Lie algebra of ADE type, then W acts transitively on Σ, so m is a constant function.

For each reflection $s \in \Sigma$, choose $\alpha_s \in \mathfrak{h}^* - \{0\}$ so that, for $x \in \mathfrak{h}$, $\alpha_s(sx) = -\alpha_s(x)$ (this means that the hyperplane given by the equation $\alpha_s = 0$ is the reflection hyperplane for s).

DEFINITION 1.1 ([CV1, CV2]). A polynomial $q \in \mathbf{C}[\mathfrak{h}]$ is said to be *m-quasi-invariant* with respect to W if, for any $s \in \Sigma$, the polynomial $q(x) - q(sx)$ is divisible by $\alpha_s(x)^{2m_s+1}$.

We will denote by Q_m the space of m-quasi-invariant polynomials with respect to W.

Notice that every element of $\mathbf{C}[\mathfrak{h}]$ is a 0-quasi-invariant, and that every W-invariant is an m-quasi-invariant for any m. Indeed if $q \in \mathbf{C}[\mathfrak{h}]^W$, then we have $q(x) - q(sx) = 0$ for all $s \in \Sigma$, and 0 is divisible by all powers of $\alpha_s(x)$. Thus in a way, $\mathbf{C}[\mathfrak{h}]^W$ can be viewed as the set of ∞-quasi-invariants.

EXAMPLE 1.2. The group $W = \mathbf{Z}/2$ acts on $\mathfrak{h} = \mathbf{C}$ by $s(v) = -v$. In this case m is a non negative integer and $\Sigma = \{s\}$. So this definition says that q is in Q_m iff $q(x) - q(-x)$ is divisible by x^{2m+1}. It is very easy to write a basis of Q_m. It is given by the polynomials $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$.

1.2 ELEMENTARY PROPERTIES OF Q_m

Some elementary properties of Q_m are collected in the following proposition.

PROPOSITION 1.3 (see [FV] and references therein).

1) $\mathbf{C}[\mathfrak{h}]^W \subset Q_m \subseteq \mathbf{C}[\mathfrak{h}]$, $Q_0 = \mathbf{C}[\mathfrak{h}]$, $Q_m \subset Q_{m'}$ *if* $m \geq m'$, $\bigcap_m Q_m = \mathbf{C}[\mathfrak{h}]^W$.
2) Q_m *is a graded subalgebra of* $\mathbf{C}[\mathfrak{h}]$.
3) *The fraction field of* Q_m *is equal to* $\mathbf{C}(\mathfrak{h})$.
4) Q_m *is a finite* $\mathbf{C}[\mathfrak{h}]^W$*-module and a finitely generated algebra.* $\mathbf{C}[\mathfrak{h}]$ *is a finite* Q_m*-module.*

Proof. 1) is immediate and has already been mentioned in 1.1.

2) Clearly Q_m is closed under addition. Let $p, q \in Q_m$. Let $s \in \Sigma$. Then

$$p(x)q(x) - p(sx)q(sx) = (p(x) - p(sx))q(x) + p(sx)(q(x) - q(sx)).$$

Since both $p(x) - p(sx)$ and $q(x) - q(sx)$ are divisible by $\alpha_s^{2m_s+1}$, we deduce that $p(x)q(x) - p(sx)q(sx)$ is also divisible by $\alpha_s^{2m_s+1}$, proving the claim.

3) Consider the polynomial

$$\delta_{2m+1}(x) = \prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}.$$

This polynomial is uniquely defined up to scaling. One has $\delta_{2m+1}(sx) = -\delta_{2m+1}(x)$ for each $s \in \Sigma$, hence $\delta_{2m+1} \in Q_m$. Take $f(x) \in \mathbf{C}[\mathfrak{h}]$. We claim that $f(x)\delta_{2m+1}(x) \in Q_m$. As a matter of fact,

$$f(x)\delta_{2m+1}(x) - f(sx)\delta_{2m+1}(sx) = (f(x) + f(sx))\delta_{2m+1}(x),$$

and by its definition $\delta_{2m+1}(x)$ is divisible by $\alpha_s(x)^{2m_s+1}$ for all $s \in \Sigma$. This implies 3).

4) By Hilbert's theorem on the finiteness of invariants, we get that $\mathbf{C}[\mathfrak{h}]^W$ is a finitely generated algebra over $\mathbf{C}$ and $\mathbf{C}[\mathfrak{h}]$ is a finite $\mathbf{C}[\mathfrak{h}]^W$-module and hence a finite Q_m-module, proving the second part of 4).

Now $Q_m \subset \mathbf{C}[\mathfrak{h}]$ is a submodule of the finite module $\mathbf{C}[\mathfrak{h}]$ over the Noetherian ring $\mathbf{C}[\mathfrak{h}]^W$. Hence it is finite. This immediately implies that Q_m is a finitely generated algebra over $\mathbf{C}$. $\square$