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REMARK. In fact, since W is a finite Coxeter group, a celebrated result of Chevalley says that the algebra $\mathbf{C}[\mathfrak{h}]^W$ is not only a finitely generated \mathbf{C} -algebra but actually a free (=polynomial) algebra. Namely, it is of the form $\mathbf{C}[q_1, \dots, q_n]$, where the q_i are homogeneous polynomials of some degrees d_i . Furthermore, if we denote by H the subspace of $\mathbf{C}[\mathfrak{h}]$ of harmonic polynomials, i.e. of polynomials killed by W -invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \rightarrow \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of $\mathbf{C}[\mathfrak{h}]^W$ - and of W -modules. In particular, $\mathbf{C}[\mathfrak{h}]$ is a free $\mathbf{C}[\mathfrak{h}]^W$ -module of rank $|W|$.

1.3 THE VARIETY X_m AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety $X_m = \text{Spec}(Q_m)$. The inclusion $Q_m \subset \mathbf{C}[\mathfrak{h}]$ induces a morphism

$$\pi: \mathfrak{h} \rightarrow X_m,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that X_m is singular for all $m \neq 0$.)

In fact, not only is π birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]). *π is a bijection.*

Proof. By the above remarks, we only have to show that π is injective. In order to achieve this, we need to prove that quasi-invariants separate points of \mathfrak{h} , i.e. that if $z, y \in \mathfrak{h}$ and $z \neq y$, then there exists $p \in Q_m$ such that $p(z) \neq p(y)$. This is obtained in the following way. Let $W_z \subset W$ be the stabilizer of z and choose $f \in \mathbf{C}[\mathfrak{h}]$ such that $f(z) \neq 0$, $f(y) = 0$. Set

$$p(x) = \prod_{s \in \Sigma, sz \neq z} \alpha_s(x)^{2m_s+1} \prod_{w \in W_z} f(wx).$$

We claim that $p(x) \in Q_m$. Indeed, let $s \in \Sigma$ and assume that $s(z) \neq z$.

We have by definition $p(x) = \alpha_s(x)^{2m_s+1} \tilde{p}(x)$, with $\tilde{p}(x)$ a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1} \tilde{p}(x) - \alpha_s(sx)^{2m_s+1} \tilde{p}(sx) = \alpha_s(x)^{2m_s+1} (\tilde{p}(x) + \tilde{p}(sx)).$$

If on the other hand, $sz = z$, i.e. $s \in W_z$, then s preserves the set $W \setminus W_z$, and hence preserves $\prod_{s \in \Sigma \cap (W \setminus W_z)} \alpha_s(x)^{2m_s+1}$ (as it acts by -1 on the products $\prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}$ and $\prod_{s \in \Sigma \cap W_z} \alpha_s(x)^{2m_s+1}$). Since $\prod_{w \in W_z} f(wx)$ is

W_z -invariant, we deduce that $p(x) - p(sx) = 0$, so that in this case $p(x) - p(sx)$ also is divisible by $\alpha_s(x)^{2m_s+1}$.

To conclude, notice that $p(z) \neq 0$. Indeed, for a reflection s , α_s vanishes exactly on the fixed points of s , so that $\prod_{s \in \Sigma, sz \neq z} \alpha_s(z)^{2m_s+1} \neq 0$. Also for all $w \in W_z$ $f(wz) = f(z) \neq 0$. On the other hand, it is clear that $p(y) = 0$. \square

EXAMPLE 1.5. Take $W = \mathbf{Z}/2$. As we have already seen, Q_m has a basis given by the monomials $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$. From this we deduce that setting $z = x^2$ and $y = x^{2m+1}$, $Q_m = \mathbf{C}[y, z]/(y^2 - z^{2m+1}) = \mathbf{C}[K]$, where K is the plane curve with a cusp at the origin, given by the equation $y^2 = z^{2m+1}$. The map $\pi: \mathbf{C} \rightarrow K$ is given by $\pi(t) = (t^{2m+1}, t^2)$, which is clearly bijective.

1.4 FURTHER PROPERTIES OF X_m

Let us get to some deeper properties of quasi-invariants. Let X be an irreducible affine variety over \mathbf{C} and $A = \mathbf{C}[X]$. Recall that, by the Noether Normalization Lemma, there exist $f_1, \dots, f_n \in \mathbf{C}[X]$ which are algebraically independent over \mathbf{C} and such that $\mathbf{C}[X]$ is a finite module over the polynomial ring $\mathbf{C}[f_1, \dots, f_n]$. This means that we have a finite morphism of X onto an affine space.

DEFINITION 1.6. A (and X) is said to be *Cohen-Macaulay* if there exist f_1, \dots, f_n as above, with the property that $\mathbf{C}[X]$ is a locally free module over $\mathbf{C}[f_1, \dots, f_n]$. (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that A is a free module.)

REMARK. If A is Cohen-Macaulay, then for any f_1, \dots, f_n which are algebraically independent over \mathbf{C} and such that A is a finite module over the polynomial ring $\mathbf{C}[f_1, \dots, f_n]$, we have that A is a locally free $\mathbf{C}[f_1, \dots, f_n]$ -module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]). Q_m is Cohen-Macaulay.

Notice that, using Chevalley's result that $\mathbf{C}[\mathfrak{h}]^W$ is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]). Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$ -module.