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Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA

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Kapitel: 1.5 The Poincaré séries of Q_m

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We show how one can prove this Theorem in 3.10. This proof follows [BEG] (the original proof of [EG2] is shorter but somewhat less conceptual). The main idea of the proof is to show that the $\mathbf{C}[\mathfrak{h}]^W$-module Q_m can be extended to a module over a bigger (noncommutative) algebra, namely the spherical subalgebra of the rational Cherednik algebra. Furthermore, this module belongs to an appropriate category of representations of this algebra, called category $\mathcal{O}$. On the other hand, it can be shown that any module over the spherical subalgebra that belongs to this category is free when restricted to the commutative algebra $\mathbf{C}[\mathfrak{h}]^W$.

1.5 The Poincaré series of Q_m

Consider now the Poincaré series

$$h_{Q_m}(t) = \sum_{r \geq 0} \dim Q_m[r] t^r ,$$

where $Q_m[r]$ denotes the graded component of Q_m of degree r. For every irreducible representation $\tau \in \widehat{W}$, define

$$\chi_\tau(t) = \sum_{r \geq 0} \dim \mathrm{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}][r]) t^r .$$

Consider the element in the group ring $\mathbf{Z}[W]$

$$\mu_m = \sum_{s \in \Sigma} m_s (1 - s) .$$

The W-invariance of m implies that μ_m lies in the center of $\mathbf{Z}[W]$. Hence it is clear that μ_m acts as a scalar, $\xi_m(\tau)$, on τ. Let d_τ be the degree of τ.

Lemma 1.9. *The scalar $\xi_m(\tau)$ is an integer.*

Proof. $\mathbf{Z}[W]$ and hence also its center, is a finite $\mathbf{Z}$-module. This clearly implies that $\xi_m(\tau)$ is an algebraic integer. Thus to prove that $\xi_m(\tau)$ is an integer, it suffices to see that $\xi_m(\tau)$ is a rational number. Let $d_{\tau,s}$ be the dimension of the space of s-invariants in τ. Taking traces we get

$$d_\tau \xi_m(\tau) = \sum_{s \in \Sigma} 2 m_s (d_\tau - d_{\tau,s}) ,$$

which gives the rationality of $\xi_m(\tau)$. □

THEOREM 1.10. *One has*

(1)
$$h_{Q_m}(t) = \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\tau)} \chi_\tau(t)\,.$$

REMARK. This theorem was proved in [FeV] modulo Theorem 1.7 (conjectured in [FV]) using the so-called Matsuo-Cherednik correspondence (see [FeV] for details). Thus, Theorem 1.10 follows from [FeV] and [EG2]. Another proof of this theorem is given in [BEG]; this is the proof we will discuss below (in Lecture 3).

EXAMPLE 1.11. If $m = 0$, since $Q_0 = \mathbf{C}[\mathfrak{h}]$, the theorem says that

$$h_{Q_0}(t) = \frac{1}{(1-t)^n} = \sum_{\tau \in \widehat{W}} d_\tau \chi_\tau(t)\,.$$

Indeed, as a W-module one has

$$\mathbf{C}[\mathfrak{h}] = \oplus_\tau \tau \otimes \mathrm{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}])\,.$$

EXAMPLE 1.12. If $W = \mathbf{Z}/2$, then $\widehat{W} = \{+,-\}$, where $+$ (respectively $-$) denotes the trivial (respectively the sign) representation. One has

$$\mathbf{C}[x] = \mathbf{C}[x^2] \oplus \mathbf{C}[x^2]x\,,$$

where $\mathbf{C}[x^2] = \mathbf{C}[x]^W$ and $\mathbf{C}[x^2]x$ is the isotypic component of the sign representation. Thus

$$\chi_+(t) = \frac{1}{1-t^2}\,, \quad \chi_-(t) = \frac{t}{1-t^2}\,,$$

$\mu_m = m(1-s)$. Thus $\xi_m(+) = 0$, $\xi_m(-) = 2m$. We deduce that

$$h_{Q_m}(t) = \frac{1}{1-t^2} + \frac{t^{2m+1}}{1-t^2}\,,$$

as we already know.

Recall now that as a graded W-module $\mathbf{C}[\mathfrak{h}]$ is isomorphic to $\mathbf{C}[\mathfrak{h}]^W \otimes H$, H being the space of harmonic polynomials. We deduce that the τ-isotypic component in $\mathbf{C}[\mathfrak{h}]$ is isomorphic to $\mathbf{C}[\mathfrak{h}]^W \otimes H_\tau$.

Set $K_\tau(t) = \sum_{r\geq 0} \dim \operatorname{Hom}_W(\tau, H[r])t^r$. This is a polynomial, called the Kostka polynomial relative to τ. We deduce that

$$\chi_\tau(t) = \frac{K_\tau(t)}{\prod_{i=1}^n (1-t^{d_i})}. \tag{2}$$

Also, if $\tau' = \tau\otimes\varepsilon$, ε being the sign representation, one has

$$K_{\tau'}(t) = K_\tau(t^{-1})t^{|\Sigma|}.$$

Set now

$$P_m(t) = \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\tau)} K_\tau(t).$$

We have

PROPOSITION 1.13 ([FeV]).

$$h_{Q_m}(t) = \frac{P_m(t)}{\prod_{i=1}^n (1-t^{d_i})}.$$

Furthermore $P_m(t) = t^{\xi_m(\varepsilon)+|\Sigma|}P_m(t^{-1})$.

Proof. Substituting the expression (2) for $\chi_\tau(t)$ in (1.10) and using the definition of $P_m(t)$, we get

$$h_{Q_m}(t) = \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\tau)} \frac{K_\tau(t)}{\prod_{i=1}^n (1-t^{d_i})} = \frac{P_m(t)}{\prod_{i=1}^n (1-t^{d_i})},$$

as desired.

Now notice that

$$\xi_m(\tau) + \xi_m(\tau') = \sum_{s\in\Sigma} 2m_s = \xi_m(\varepsilon).$$

Using this we get

$$\begin{aligned} t^{\xi_m(\varepsilon)+|\Sigma|}P_m(t^{-1}) &= \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\varepsilon)-\xi_m(\tau)} t^{|\Sigma|} K_\tau(t^{-1}) \\ &= \sum_{\tau'\in\widehat{W}} d_{\tau'} t^{\xi_m(\tau')} K_{\tau'}(t) = P_m(t), \end{aligned}$$

as desired. □

From this we deduce

THEOREM 1.14 ([EG2, BEG, FeV], conjectured in [FV]). *The ring Q_m of m-quasi-invariants is Gorenstein.*

Proof. By Stanley's theorem (see [Eis]), a positively graded Cohen-Macaulay domain A is Gorenstein iff its Poincaré series is a rational function $h(t)$ satisfying the equation $h(t^{-1}) = (-1)^n t^l h(t)$, where l is an integer and n is the dimension of the spectrum of A. Thus the result follows immediately from Proposition 1.13. □

1.6 THE RING OF DIFFERENTIAL OPERATORS ON X_m

Finally, let us introduce the ring $\mathcal{D}(X_m)$ of differential operators on X_m, that is the ring of differential operators with coefficients in $\mathbf{C}(\mathfrak{h})$ mapping Q_m to Q_m. It is clear that this definition coincides with Grothendieck's well-known definition ([Bj]).

THEOREM 1.15 ([BEG]). *$\mathcal{D}(X_m)$ is a simple algebra.*

REMARK 1.16. a) The ring of differential operators on a smooth affine algebraic variety is always simple (see [Bj], Chapter 3).

b) By a result of M. van den Bergh [VdB], for a non-smooth variety, the simplicity of the ring of differential operators implies the Cohen-Macaulay property of this variety.

2. LECTURE 2

We will now see how the ring Q_m appears in the theory of completely integrable systems.

2.1 HAMILTONIAN MECHANICS AND INTEGRABLE SYSTEMS

Recall the basic setup of Hamiltonian mechanics [Ar]. Consider a mechanical system with configuration space X (a smooth manifold). Then the phase space of this system is T^*X, the cotangent bundle on X. The space T^*X is naturally a symplectic manifold, and in particular we have an operation of Poisson bracket on functions on T^*X. A point of T^*X is a pair (x,p), where $x \in X$ is the position and $p \in T_x^*X$ is the momentum. Such pairs are