3.1 Shift operator and construction of the Baker-Akhiezer function

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

The solution can easily be computed by differentiating the first equation and then subtracting the second, thus obtaining the new system

$$\begin{cases} \psi'' - \frac{2}{x}\psi' = k^2\psi\,, \\ \psi'' - (\frac{1}{x} + k^2 x)\psi' = -k^3 x\psi\,. \end{cases}$$

Taking the difference, we get the first order equation

$$\psi' = \frac{k^2 x}{kx-1}\psi\,,$$

whose solution (up to constants) is given by $\psi = (kx-1)\,e^{kx}$.

In fact, one can easily calculate ψ_m for a general m.

PROPOSITION 2.12. $\psi_m(k,x) = (x\partial - 2m+1)(x\partial - 2m-1)\cdots(x\partial - 1)\,e^{kx}$.

Proof. We could use the direct method of Example 2.11, but it is more convenient to proceed differently. Namely, we have

$$(\partial^2 - \frac{2m}{x}\partial)(x\partial - 2m+1) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)$$

as it is easy to verify directly. So using induction on m starting with $m=0$, we get

$$(\partial^2 - \frac{2m}{x}\partial)\psi_m(k,x) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)\psi_{m-1}(k,x) = k^2\psi_m(k,x)\,,$$

and $\psi_m(k,x)$ is our solution. $\square$

3. LECTURE 3

3.1 SHIFT OPERATOR AND CONSTRUCTION OF THE BAKER-AKHIEZER FUNCTION

In Lecture 2, we have introduced the Baker-Akhiezer function $\psi(k,x)$ for the operator

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)}\partial_{\alpha_s}\,.$$

The way to construct $\psi(k,x)$ is via the Opdam shift operator. Given a function $m\colon \Sigma \to \mathbf{Z}_+$, Opdam showed in [Op1] that there exists a unique W-invariant differential operator S_m of the form $\delta_m(x)\delta_m(\partial_x)+l.o.t.$, with $\delta_m(x) = \prod_{s\in\Sigma} \alpha_s^{m_s}$ such that

$$L_q S_m = S_m q(\partial)$$

for every $q \in \mathbf{C}[\mathfrak{h}] = \mathbf{C}[q_1,\dots,q_n]$. From this, if we set $\psi(k,x) = S_m e^{(k,x)}$, we get

$$L_q\psi = S_m q(\partial)\, e^{(k,x)} = q(k)\psi\,, \tag{7}$$

$q \in \mathbf{C}[q_1,\dots,q_n]$.

We claim that equation (7) must in fact hold for all $q \in Q_m$. Indeed, near a generic point x, the functions $\psi(wk,x)$ are obviously linearly independent and satisfy (7) for symmetric q. Thus, they are a basis in the space of solutions (we know that this space is $|W|$-dimensional). Consider the matrix of L_q in this basis for any $q \in Q_m$. Since $\psi(k,x)$ is a polynomial multiplied by $e^{(k,x)}$, this matrix must be diagonal with eigenvalues $q(k)$, as desired.

EXAMPLE 3.1. As we have seen in the previous section, for $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$,

$$S_m = (x\partial - 2m + 1)(x\partial - 2m - 1)\cdots(x\partial - 1)\,.$$

3.2 BEREST'S FORMULA FOR L_q

We are now going to give an explicit construction of the operators L_q for any $q \in Q_m$.

Let us identify, using our W-invariant scalar product, $\mathfrak{h}$ with $\mathfrak{h}^*$, and let us choose a orthonormal basis $x_1,\dots,x_n$ in $\mathfrak{h}^*$. If $x \in \mathfrak{h}^*$, we will write D_x for the Dunkl operator relative to the vector in $\mathfrak{h}$ corresponding to x under our identification. Thus

$$L = \sum_{i=1}^{n} D_{x_i}^2\,.$$

PROPOSITION 3.2 (Berest [Be]). *If $q \in Q_m$ is a homogeneous element of degree d, then*

$$(\operatorname{ad} L)^{d+1} q = 0\,.$$

Proof. It is enough to prove that

$$((\operatorname{ad} L)^{d+1} q)\psi(k,x) = 0\,.$$

Indeed, it follows from the definition of $\psi(k,x)$ that in the ring $\mathcal{D}(U)$ this implies: $((\operatorname{ad} L)^{d+1} q)\, S_m = 0$, so that $(\operatorname{ad} L)^{d+1}\, q = 0$, since $\mathcal{D}(U)$ is a domain.