

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
Autor: Etingof, Pavel / Strickland, Elisabetta
Kapitel: 3.5 The spherical subalgebra
DOI: <https://doi.org/10.5169/seals-66677>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 19.04.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Proof. It is easy to see that the map μ is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n} w$ are linearly independent in H_c . To do this, it suffices to show that the images of these monomials under the homomorphism ϕ , i.e. $x_1^{i_1} \dots x_n^{i_n} D_{x_1}^{j_1} \dots D_{x_n}^{j_n} w$, are linearly independent.

Given an element $A \in \mathcal{A}$, writing $A = \sum_{w \in W} P_w w$ with $P_w \in \mathcal{D}(U)$ we define the order of A , $\text{ord} A$, as the maximum of the orders of the P_w 's. Notice that $\text{ord} AB \leq \text{ord} A + \text{ord} B$. We now remark that for any sequence of non negative indices $(i_1, \dots, i_n)$,

$$D_{x_1}^{i_1} \dots D_{x_n}^{i_n} = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + l.o.t.$$

Indeed this is true for D_{x_i} . We proceed by induction on $r = i_1 + \dots + i_n$. We can clearly assume $i_1 > 0$, so by induction,

$$D_{x_1}^{i_1} \dots D_{x_n}^{i_n} = (\partial_{x_1} + l.o.t.)(\partial_{x_1}^{i_1-1} \dots \partial_{x_n}^{i_n} + l.o.t.) = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + l.o.t.$$

From this we deduce that for any pair of multiindices $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$, $w \in W$, setting $x_I = x_1^{i_1} \dots x_n^{i_n}$, $D_J = D_{x_1}^{j_1} \dots D_{x_n}^{j_n}$, $\partial_J = \partial_{x_1}^{j_1} \dots \partial_{x_n}^{j_n}$, we have

$$x_I D_J w = x_I \partial_J w + l.o.t.$$

Using this and the linear independence of the elements $x_I \partial_J w$, it is immediate to conclude that the elements $x_I D_J w$ are linearly independent, proving our claim. $\square$

REMARK 1. We see that the homomorphism ϕ identifies H_c with the subalgebra of $\mathcal{A}$ generated by $\mathbf{C}[\mathfrak{h}]$, the Dunkl operators D_y , $y \in \mathfrak{h}$ and W .

REMARK 2. Another way to state the PBW theorem is the following. Let $F^\bullet$ be a filtration on H_c defined by $\deg(x_i) = \deg(y_i) = 1$, $\deg(w) = 0$. Then we have a natural surjective mapping from $\mathbf{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$ to the associated graded algebra $\text{gr}(H_c)$. The PBW theorem claims that this map is in fact an isomorphism.

3.5 THE SPHERICAL SUBALGEBRA

Let us now introduce the idempotent

$$e = \frac{1}{W} \sum_{w \in W} w \in \mathbf{C}[W].$$

DEFINITION 3.13. The *spherical subalgebra* of H_c is the algebra eH_ce .

Notice that $1 \notin eH_ce$. On the other hand, since $ex = xe = e$ for $x \in eH_ce$, e is the unit for the spherical subalgebra. We can embed both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ in the spherical subalgebra as follows. Take $f \in \mathbf{C}[\mathfrak{h}^*]^W$ (the other case is identical) and set $m_e(f) = fe$. Since f is invariant, we have $efe = fe^2 = fe = m_e(f)$, so that m_e actually maps $\mathbf{C}[\mathfrak{h}^*]^W$ to eH_ce . The injectivity is clear from the PBW-theorem. As for the fact that m_e is a homomorphism, we have $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$. From now on, we will consider both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ as subalgebras of the spherical subalgebra.

3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras H_c and eH_ce .

DEFINITION 3.14. The category $\mathcal{O}(H_c)$ (resp. $\mathcal{O}(eH_ce)$) is the full subcategory of the category of H_c -modules (resp. eH_ce -modules) whose objects are the modules M such that

- 1) M is finitely generated.
- 2) For all $v \in M$, the subspace $\mathbf{C}[\mathfrak{h}^*]^W v \subset M$ is finite dimensional.

We can define a functor

$$F: \mathcal{O}(H_c) \rightarrow \mathcal{O}(eH_ce)$$

by setting $F(M) = eM$. It is easy to show that $F(M)$ is an object of $\mathcal{O}(eH_ce)$.

We are now going to explain how to construct some modules in $\mathcal{O}(H_c)$ which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take $\lambda \in \mathfrak{h}^*$. Denote by $W_\lambda \subset W$ the stabilizer of λ . Take an irreducible W_λ -module τ . We define a structure of $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$ -module on τ by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau, w \in W_\lambda, f \in \mathbf{C}[\mathfrak{h}^*].$$

It is easy to see that this action is well defined and we denote this module by $\lambda \# \tau$. We can then consider the H_c -module

$$M(\lambda, \tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]} \lambda \# \tau.$$

This is called a Whittaker module. In the special case $\lambda = 0$ (and hence $W_\lambda = W$), the module $M(0, \tau)$ is called a Verma module. It is clear that these are objects of $\mathcal{O}$. Notice that as $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$ -module, $M(\lambda, \tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$.