Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 49 (2003)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ATIYAH'S L^2-INDEX THEOREM

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Kapitel: 3. Hilbert modules

DOI: https://doi.org/10.5169/seals-66679

3. Hilbert modules

Recall that for $H < G$ and X an H-space, the *induced* G-space is

$$G \times_H X = (G \times X)/H$$

where H acts on $G \times X$ via $h \cdot (g,x) = (gh^{-1}, hx)$ and the left G-action on $G \times_H X$ is given by $g \cdot [k,x] = [gk,x]$ (where $[k,x]$ denotes the class of the pair $(k,x) \in G \times X$ in $G \times_H X$). For $A \subseteq \ell^2(H)^n$ a Hilbert H-module one defines $\mathrm{Ind}_H^G(A)$, the *induced* Hilbert G-module, as follows:

$$\mathrm{Ind}_H^G(A) = \left\{ f \colon G \to A, \quad f(gh) = h^{-1}f(g), \quad \sum_{\gamma \in G/H} \|f(\gamma)\|^2 < \infty \right\}.$$

On $\mathrm{Ind}_H^G(A)$ the action of G is given as follows:

$$(\gamma \cdot f)(\mu) = f(\gamma^{-1}\mu), \qquad \gamma, \mu \in G \text{ and } f \in \mathrm{Ind}_H^G(A).$$

For $\widetilde{M}$ an H-free, cocompact Riemannian manifold and $\widetilde{D}$ an H-equivariant pseudo-differential operator on $\widetilde{M}$, one can express the lift $\overline{D}$ of $\widetilde{D}$ to $\overline{M} = G \times_H \widetilde{M}$ as follows. Fix a set R of representatives for G/H and write $\pi \colon \overline{M} \to \widetilde{M}$ for the projection; a section $\overline{s} \in C_c^\infty(\overline{M}, \pi^*E)$ is a collection

$$\overline{s} = \{\widetilde{s}_r\}_{r \in R},$$

where $\widetilde{s}_r \in C_c^\infty(\widetilde{M}, E)$ is the zero section for all but finitely many r's, and $\overline{s}([g,\widetilde{m}]) = \widetilde{s}_r(h\widetilde{m})$, if $[r, h\widetilde{m}] = [g, \widetilde{m}] \in G \times_H \widetilde{M}$. Now the lift $\overline{D}$ of $\widetilde{D}$ to $\overline{M} = G \times_H \widetilde{M}$ satisfies

$$\overline{D}\,\overline{s} = \left\{\widetilde{D}\,\widetilde{s}_r\right\}_{r \in R}.$$

LEMMA 3.1. *Let M be a closed Riemannian manifold, D a pseudo-differential operator on M and $\widetilde{M}$ a regular cover of M with countable transformation group H. Consider an inclusion $H < G$ and form the regular cover $\overline{M} = G \times_H \widetilde{M}$ of M. Then for the lifts $\widetilde{D}$ of D to $\widetilde{M}$ and $\overline{D}$ of $\widetilde{D}$ to $\overline{M}$,*

$$\mathrm{Index}_H(\widetilde{D}) = \mathrm{Index}_G(\overline{D}).$$

Proof. It is enough to see that $S_{\overline{D}} \cong \mathrm{Ind}_H^G(S_{\widetilde{D}})$. Indeed, it is well-known (see [9]) that for a Hilbert H-module A one has

$$\dim_H(A) = \dim_G(\mathrm{Ind}_H^G(A)).$$

For R a fixed set of representatives for G/H, the map

$$\begin{aligned}\varphi_R\colon \operatorname{Ind}_H^G(S_{\widetilde{D}}) &\to S_{\bar{D}}\\ f &\mapsto \{f(r)\}_{r\in R}\end{aligned}$$

is well-defined by H-equivariance of the elements of $S_{\widetilde{D}}$ and one checks that it defines a G-equivariant isometric bijection. Similarly for the adjoint operators.

The following example is a particular case of the previous lemma.

EXAMPLE 3.2. Let us look at the case $\widetilde{M} = M \times G$. A section $\widetilde{s} \in C_c^\infty(\widetilde{M}, \pi^*E)$ is an element $\widetilde{s} = \{s_g\}_{g\in G}$ where $s_g \in C^\infty(M,E)$ and $s_g = 0$ for all but finitely many g's. Note that $L^2(\widetilde{M}, \pi^*E)$ can be identified with $\ell^2(G) \otimes L^2(M,E)$. Now

$$\widetilde{D}\widetilde{s} = \{Ds_g\}_{g\in G} \in C_c^\infty(\widetilde{M}, \pi^*F)$$

and hence $S_{\widetilde{D}}$ may be identified with $\ell^2(G) \otimes S_D \cong \ell^2(G)^d$, where $d = \dim_{\mathbf{C}}(S_D)$. In this identification the projection P onto $S_{\widetilde{D}}$ becomes the identity in $M_d(\mathcal{N}(G))$ and thus

$$\dim_G(S_{\widetilde{D}}) = \sum_{i=1}^{d} \langle e, e\rangle = d = \dim_{\mathbf{C}}(S_D)\,.$$

A similar argument for D^* shows that in this case not only does the L^2-Index of $\widetilde{D}$ coincide with the Index of D, but also the individual terms of the difference correspond to each other. This is not the case in general, see Example 2.2.

4. ON K-HOMOLOGY

Many ideas of this section go back to the seminal article by Baum and Connes [3], which has been circulating for many years and has only recently been published.

An elliptic pseudo-differential operator D on the closed manifold M can also be used to define an element $[D] \in K_0(M)$, the K-homology of M, and according to Baum and Douglas [4], all elements of $K_0(M)$ are of the form $[D]$. The index defined in Section 2 extends to a well-defined