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3. HILBERT MODULES

Recall that for $H < G$ and X an H -space, the *induced* G -space is

$$G \times_H X = (G \times X)/H$$

where H acts on $G \times X$ via $h \cdot (g, x) = (gh^{-1}, hx)$ and the left G -action on $G \times_H X$ is given by $g \cdot [k, x] = [gk, x]$ (where $[k, x]$ denotes the class of the pair $(k, x) \in G \times X$ in $G \times_H X$). For $A \subseteq \ell^2(H)^n$ a Hilbert H -module one defines $\text{Ind}_H^G(A)$, the *induced* Hilbert G -module, as follows:

$$\text{Ind}_H^G(A) = \left\{ f: G \rightarrow A, \quad f(gh) = h^{-1}f(g), \quad \sum_{\gamma \in G/H} \|f(\gamma)\|^2 < \infty \right\}.$$

On $\text{Ind}_H^G(A)$ the action of G is given as follows:

$$(\gamma \cdot f)(\mu) = f(\gamma^{-1}\mu), \quad \gamma, \mu \in G \quad \text{and} \quad f \in \text{Ind}_H^G(A).$$

For \tilde{M} an H -free, cocompact Riemannian manifold and \tilde{D} an H -equivariant pseudo-differential operator on \tilde{M} , one can express the lift \bar{D} of \tilde{D} to $\bar{M} = G \times_H \tilde{M}$ as follows. Fix a set R of representatives for G/H and write $\pi: \bar{M} \rightarrow \tilde{M}$ for the projection; a section $\bar{s} \in C_c^\infty(\bar{M}, \pi^*E)$ is a collection

$$\bar{s} = \{\tilde{s}_r\}_{r \in R},$$

where $\tilde{s}_r \in C_c^\infty(\tilde{M}, E)$ is the zero section for all but finitely many r 's, and $\bar{s}([g, \tilde{m}]) = \tilde{s}_r(h\tilde{m})$, if $[r, h\tilde{m}] = [g, \tilde{m}] \in G \times_H \tilde{M}$. Now the lift \bar{D} of \tilde{D} to $\bar{M} = G \times_H \tilde{M}$ satisfies

$$\bar{D}\bar{s} = \left\{ \tilde{D}\tilde{s}_r \right\}_{r \in R}.$$

LEMMA 3.1. *Let M be a closed Riemannian manifold, D a pseudo-differential operator on M and \tilde{M} a regular cover of M with countable transformation group H . Consider an inclusion $H < G$ and form the regular cover $\bar{M} = G \times_H \tilde{M}$ of M . Then for the lifts \tilde{D} of D to \tilde{M} and \bar{D} of \tilde{D} to \bar{M} ,*

$$\text{Index}_H(\tilde{D}) = \text{Index}_G(\bar{D}).$$

Proof. It is enough to see that $S_{\bar{D}} \cong \text{Ind}_H^G(S_{\tilde{D}})$. Indeed, it is well-known (see [9]) that for a Hilbert H -module A one has

$$\dim_H(A) = \dim_G(\text{Ind}_H^G(A)).$$

For R a fixed set of representatives for G/H , the map

$$\begin{aligned} \varphi_R: \text{Ind}_H^G(S_{\tilde{D}}) &\rightarrow S_{\tilde{D}} \\ f &\mapsto \{f(r)\}_{r \in R} \end{aligned}$$

is well-defined by H -equivariance of the elements of $S_{\tilde{D}}$ and one checks that it defines a G -equivariant isometric bijection. Similarly for the adjoint operators.

The following example is a particular case of the previous lemma.

EXAMPLE 3.2. Let us look at the case $\tilde{M} = M \times G$. A section $\tilde{s} \in C_c^\infty(\tilde{M}, \pi^*E)$ is an element $\tilde{s} = \{s_g\}_{g \in G}$ where $s_g \in C^\infty(M, E)$ and $s_g = 0$ for all but finitely many g 's. Note that $L^2(\tilde{M}, \pi^*E)$ can be identified with $\ell^2(G) \otimes L^2(M, E)$. Now

$$\tilde{D}\tilde{s} = \{Ds_g\}_{g \in G} \in C_c^\infty(\tilde{M}, \pi^*F)$$

and hence $S_{\tilde{D}}$ may be identified with $\ell^2(G) \otimes S_D \cong \ell^2(G)^d$, where $d = \dim_{\mathbb{C}}(S_D)$. In this identification the projection P onto $S_{\tilde{D}}$ becomes the identity in $M_d(\mathcal{N}(G))$ and thus

$$\dim_G(S_{\tilde{D}}) = \sum_{i=1}^d \langle e, e \rangle = d = \dim_{\mathbb{C}}(S_D).$$

A similar argument for D^* shows that in this case not only does the L^2 -Index of \tilde{D} coincide with the Index of D , but also the individual terms of the difference correspond to each other. This is not the case in general, see Example 2.2.

4. ON K -HOMOLOGY

Many ideas of this section go back to the seminal article by Baum and Connes [3], which has been circulating for many years and has only recently been published.

An elliptic pseudo-differential operator D on the closed manifold M can also be used to define an element $[D] \in K_0(M)$, the K -homology of M , and according to Baum and Douglas [4], all elements of $K_0(M)$ are of the form $[D]$. The index defined in Section 2 extends to a well-defined