Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 49 (2003)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ADDITIVE NUMBER THEORY SHEDS EXTRA LIGHT ON THE HOPF-STIEFEL o FUNCTION

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DOI: https://doi.org/10.5169/seals-66682

3. From Theorem 3 to Theorem 1

We first use the theorem of Eliahou and Kervaire (see Section 3 of [5]), which states that if p is an arbitrary prime, r and s two integers, then

$$\beta_p(r,s) = \mu_{(\mathbf{Z}/p\mathbf{Z})^d}(r,s) \tag{3.1}$$

whenever $p^d \geq r, s$.

Now, from Theorem 10 of [1], it follows that μ_G coincides with $\mu_{G'}$ as soon as G and G' are two Abelian p-groups of the same order. In other words,

$$\mu_{(\mathbf{Z}/p\mathbf{Z})^d}(r,s) = \mu_{\mathbf{Z}/p^d\mathbf{Z}}(r,s)\,. \tag{3.2}$$

We would like to emphasize that from our method (more precisely, using simply Lemma 1) together with an inductive argument (the quotient groups of $(\mathbf{Z}/p\mathbf{Z})^d$ have the same form), we are able to derive a simple direct (that is, without using [1]) alternative proof of (3.2). Indeed, the only thing to verify is that if

$$r+s-1 < \left(\lceil r/p^k\rceil + \lceil s/p^k\rceil - 1\right)p^k \tag{3.3}$$

for any $k \geq 1$ then we can construct sets $\mathcal{A}$ and $\mathcal{B}$ of respective cardinalities r and s with $|\mathcal{A}+\mathcal{B}| = r+s-1$. This is achieved by taking for $\mathcal{A}$ (resp. for $\mathcal{B}$) the r (resp. the s) smallest possible elements in the sense of the lexicographic order. Hypothesis (3.3) then ensures that, in this case, $|\mathcal{A}+\mathcal{B}| = r+s-1$.

We are now ready to prove Theorem 1. We put for instance $d = r+s$ (but any sufficiently large d will do). Using consecutively (3.1), (3.2) and Theorem 3, we obtain

$$\begin{aligned}
\beta_p(r,s) &= \mu_{(\mathbf{Z}/p\mathbf{Z})^d}(r,s)\\
&= \mu_{\mathbf{Z}/p^d\mathbf{Z}}(r,s)\\
&= \min_{t\mid p^d}\left(\lceil r/t\rceil + \lceil s/t\rceil - 1\right)t\\
&= \min_{u\leq d}\left(\lceil r/p^u\rceil + \lceil s/p^u\rceil - 1\right)p^u\\
&= \min_{u\in\mathbf{N}}\left(\lceil r/p^u\rceil + \lceil s/p^u\rceil - 1\right)p^u,
\end{aligned}$$

which proves Theorem 3.

ACKNOWLEDGEMENTS. I was introduced to the μ function by Shalom Eliahou during the conference "Multilinear Algebra and Matroid Theory" held in Lisboa, Portugal (March 24–26, 2002). There, he asked me for a formula for $\mu_{\mathbf{Z}/n\mathbf{Z}}$, a question which led me to Theorem 3. He then kindly indicated to me the application to the β_p functions. The author is grateful to Shalom Eliahou for his help as well as for a careful reading of preliminary versions of this paper. I would like to associate Michel Kervaire to these thanks, for the interest he took in this paper.

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