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## 3. FROM THEOREM 3 TO THEOREM 1

We first use the theorem of Eliahou and Kervaire (see Section 3 of [5]), which states that if  $p$  is an arbitrary prime,  $r$  and  $s$  two integers, then

$$(3.1) \quad \beta_p(r, s) = \mu_{(\mathbf{Z}/p\mathbf{Z})^d}(r, s)$$

whenever  $p^d \geq r, s$ .

Now, from Theorem 10 of [1], it follows that  $\mu_G$  coincides with  $\mu_{G'}$  as soon as  $G$  and  $G'$  are two Abelian  $p$ -groups of the same order. In other words,

$$(3.2) \quad \mu_{(\mathbf{Z}/p\mathbf{Z})^d}(r, s) = \mu_{\mathbf{Z}/p^d\mathbf{Z}}(r, s).$$

We would like to emphasize that from our method (more precisely, using simply Lemma 1) together with an inductive argument (the quotient groups of  $(\mathbf{Z}/p\mathbf{Z})^d$  have the same form), we are able to derive a simple direct (that is, without using [1]) alternative proof of (3.2). Indeed, the only thing to verify is that if

$$(3.3) \quad r + s - 1 < (\lceil r/p^k \rceil + \lceil s/p^k \rceil - 1) p^k$$

for any  $k \geq 1$  then we can construct sets  $\mathcal{A}$  and  $\mathcal{B}$  of respective cardinalities  $r$  and  $s$  with  $|\mathcal{A} + \mathcal{B}| = r + s - 1$ . This is achieved by taking for  $\mathcal{A}$  (resp. for  $\mathcal{B}$ ) the  $r$  (resp. the  $s$ ) smallest possible elements in the sense of the lexicographic order. Hypothesis (3.3) then ensures that, in this case,  $|\mathcal{A} + \mathcal{B}| = r + s - 1$ .

We are now ready to prove Theorem 1. We put for instance  $d = r + s$  (but any sufficiently large  $d$  will do). Using consecutively (3.1), (3.2) and Theorem 3, we obtain

$$\begin{aligned} \beta_p(r, s) &= \mu_{(\mathbf{Z}/p\mathbf{Z})^d}(r, s) \\ &= \mu_{\mathbf{Z}/p^d\mathbf{Z}}(r, s) \\ &= \min_{t|p^d} (\lceil r/t \rceil + \lceil s/t \rceil - 1) t \\ &= \min_{u \leq d} (\lceil r/p^u \rceil + \lceil s/p^u \rceil - 1) p^u \\ &= \min_{u \in \mathbf{N}} (\lceil r/p^u \rceil + \lceil s/p^u \rceil - 1) p^u, \end{aligned}$$

which proves Theorem 3.

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