

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 49 (2003)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** NOTE ON THE HOPF-STIEFEL FUNCTION  
**Autor:** Eliahou, Shalom / Kervaire, Michel  
**Kapitel:** Introduction  
**DOI:** <https://doi.org/10.5169/seals-66683>

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## A NOTE ON THE HOPF-STIEFEL FUNCTION

by Shalom ELIAHOU\*) and Michel KERVAIRE

### INTRODUCTION

In the preceding paper of this volume [P], Alain Plagne gives a formula for the (generalized) Hopf-Stiefel function  $\beta_p$ .

Given a prime number  $p$ , and two positive integers  $r, s$ , recall that  $\beta_p(r, s)$  is defined as the smallest integer  $n$  such that  $(x + y)^n \in (x^r, y^s)$ , where  $(x^r, y^s)$  is the ideal generated by  $x^r$  and  $y^s$  in the polynomial ring  $\mathbb{F}_p[x, y]$ .

Plagne's theorem reads

**THEOREM 1.** *Let  $r, s$  be positive integers, then  $\beta_p(r, s)$  is given by the formula*

$$(1) \quad \beta_p(r, s) = \min_{t \in \mathbb{N}} \left( \left\lceil \frac{r}{p^t} \right\rceil + \left\lceil \frac{s}{p^t} \right\rceil - 1 \right) p^t.$$

In [P], this formula is derived as a corollary of a theorem on Additive Number Theory, Theorem 4, which is the main result of the paper.

Here, we give another proof of Theorem 1 using a purely arithmetical argument.

Recall from [EK, p. 22], where  $\beta_p(r, s)$  was introduced, that this function can be described in terms of the  $p$ -adic expansions of  $r - 1$  and  $s - 1$  as follows.

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\*) During the preparation of this paper, the first author has partially benefited from a research contract with the Fonds National Suisse pour la Recherche Scientifique.

THEOREM 2. Let  $r - 1 = \sum_{i \geq 0} a_i p^i$  and  $s - 1 = \sum_{i \geq 0} b_i p^i$  be the respective  $p$ -adic expansions of  $r - 1$  and  $s - 1$ , with  $0 \leq a_i, b_i \leq p - 1$  for all  $i$ .

Define the integer  $k$  as the largest index for which  $a_k + b_k \geq p$ , if any exists. Otherwise, that is if  $a_i + b_i \leq p - 1$  for all  $i \geq 0$ , set  $k = -1$ .

Then,  $\beta_p(r, s)$  is determined by

$$(2) \quad \beta_p(r, s) = \left( \left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1}.$$

Although the point of Plagne's paper is to stress the relationship of his formula with Additive Number Theory, it is interesting to note that (1) also admits a direct proof using the above Theorem 2.

This is the content of the next section. In Section 2, we provide a simple proof of Theorem 2.

## 1. DERIVING THEOREM 1 FROM THEOREM 2

It is very easy to understand the relationship of the floor-function  $\lfloor \xi \rfloor$ , or integral part of  $\xi$ , appearing in Theorem 2, with the ceiling-function  $\lceil \xi \rceil$ , the smallest integer at least as big as  $\xi$ , used in formula (1).

The main object of this section will be to locate the minimum over  $\ell \geq 0$  of the expression  $\left( \left\lfloor \frac{r}{p^\ell} \right\rfloor + \left\lfloor \frac{s}{p^\ell} \right\rfloor - 1 \right) p^\ell$  and to show that this minimum is attained at  $\ell = k + 1$  with  $k$  as defined in Theorem 2.

For every index  $\ell \geq 0$ , we have

$$0 < \frac{1 + \sum_{i=0}^{\ell-1} a_i p^i}{p^\ell} \leq \frac{1 + \sum_{i=0}^{\ell-1} (p-1) p^i}{p^\ell} = 1. \quad \cdot$$

Since  $r = 1 + \sum_{i \geq 0} a_i p^i$ , it follows that

$$\left\lfloor \frac{r}{p^\ell} \right\rfloor = \sum_{i \geq 0} a_{i+\ell} p^i + 1.$$

Similarly, we have  $0 \leq \frac{\sum_{i=0}^{\ell-1} a_i p^i}{p^\ell} \leq \frac{\sum_{i=0}^{\ell-1} (p-1) p^i}{p^\ell} = \frac{p^\ell - 1}{p^\ell} < 1$ , and

$$(3) \quad \left\lfloor \frac{r-1}{p^\ell} \right\rfloor = \sum_{i \geq 0} a_{i+\ell} p^i.$$