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constructed in a similar way. Namely, we sew in the boundary of a 2-cell along the path corresponding to each boundary word of a tile in $\mathcal{T}$. (Technically, we must sew in a cell for *every possible* boundary word, where all possible base points are considered.) Then $Y(\mathcal{T}) \to X(\mathcal{T})$ is also a normal covering map, again whose group of deck transformations is $\mathbf{Z}^2$ acting via translations of the square lattice. Moreover, the restriction to Y is the covering map $Y \to X$.

The fundamental group of $X(\mathcal{T})$ is the tile path group $P(\mathcal{T})$, and the covering space $Y(\mathcal{T})$ corresponds to the subgroup $\pi(\mathcal{T}) \subseteq P(\mathcal{T})$. The first homology group of $Y(\mathcal{T})$ is the tile homology group, $H(\mathcal{T})$. Thus Theorem 4.11 can be considered as a special case of the Hurewicz Isomorphism Theorem.

5. Strategy for working with tile path groups

We have shown above how to translate tiling problems into problems in finitely presented groups, so we might hope to be able to resolve such questions. Unfortunately, the situation is grim. The so-called *word problem*, as well as many related problems, is known to be unsolvable, which means that no algorithm can answer the question for all possible values of the input.

This is not the end of our story, for we are not trying to solve every word problem. We might hope, however optimistically, that the word problems that arise for us can be solved, whether by hook or by crook. The algorithmic unsolvability of these problems should serve to temper any optimism that we can muster.

The tile homotopy method has been successfully applied in several cases, see [2, Exercise for Experts], [4], [13], [14]. Despite these efforts, results have been found in only a handful of cases. In this section, we give a simple strategy for understanding tile homotopy groups, which allows many new cases to be handled. In view of the difficulty in working with finitely presented groups, we understand that our approach cannot be algorithmic, nor can we expect to be able to apply it in all cases. Nonetheless, we are able to use our strategy to handle numerous new cases.

The tile path group for a finite set $\mathcal{T}$ of prototiles is given by a finitely presented group. We are more interested in the tile homotopy group, which is a subgroup of infinite index. The infiniteness of this index is unfortunate, in light of the following well-known result.

PROPOSITION 5.1. *If G is a finitely generated [respectively, finitely presented] group, and $H \subseteq G$ is a subgroup of finite index, then H is also finitely generated [respectively, finitely presented].* □

The usual proof uses covering space theory, similar to the determination of the group, C, of closed paths above. Moreover, in the finitely presented case, a presentation of H can be computed explicitly. We will do this later, with the help of the computer software package GAP [5]. There is plenty of interesting combinatorial group theory involved in this, but it is well understood, so it is not our place to discuss it here.

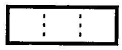
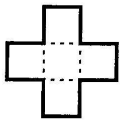
If the index $(G : H)$ is not finite, then H can fail to be finitely generated. A typical example exhibiting this behavior is the case $C \subseteq P$ that we saw earlier.

In general, the tile homotopy group will not be finitely generated. However, in some special cases, it will be. The method of demonstrating this is a non-abelian analogue of the technique for showing finite generation of the tile homology group, as in Examples 2.5 and 2.7. In order to achieve this, we need to find some relations in the tile path group.

THEOREM 5.2. *Suppose that $\bar{x}^m$ and $\bar{y}^n$ are central in $P(\mathcal{T})$, for some positive m and n. Then the natural map $P(\mathcal{T}) \twoheadrightarrow \widetilde{P}(\mathcal{T}) = P(\mathcal{T})/\langle \bar{x}^m, \bar{y}^n \rangle$ induces an isomorphism of $\pi(\mathcal{T})$ onto its image, $\widetilde{\pi}(\mathcal{T})$. Moreover, $\widetilde{\pi}(\mathcal{T})$ has index mn inside $\widetilde{P}(\mathcal{T})$ and it is generated by the images of the elements $\bar{c}_{ij} = \bar{x}^i \bar{y}^j \bar{x} \bar{y} \bar{x}^{-1} \bar{y}^{-1} \bar{y}^{-j} \bar{x}^{-i}$ for $0 \leq i < m$ and $0 \leq j < n$.*

Proof. Note that $\pi(\mathcal{T})$ is normal in $P(\mathcal{T})$, with quotient $P(\mathcal{T})/\pi(\mathcal{T}) \cong P/C \cong \mathbf{Z}^2$. This quotient is the group of translations of the grid, so $\bar{x}$ and $\bar{y}$ map to rightward and upward translation by 1 unit each. Let $N = \langle \bar{x}^m, \bar{y}^n \rangle \subseteq P(\mathcal{T})$, which, by hypothesis, is central in $P(\mathcal{T})$. Now N maps injectively to $P(\mathcal{T})/\pi(\mathcal{T})$, whence N and $\pi(\mathcal{T})$ intersect trivially. Thus $\pi(\mathcal{T})$ maps injectively to $P(\mathcal{T})/N = \widetilde{P}(\mathcal{T})$. This proves the first statement. Next, note that $\widetilde{P}(\mathcal{T})/\widetilde{\pi}(\mathcal{T}) \cong P(\mathcal{T})/N\pi(\mathcal{T}) \cong \mathbf{Z}^2/\langle \text{image of } \bar{x}^m, \bar{y}^n \rangle \cong (\mathbf{Z}/m\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$. This shows that the index $(\widetilde{P}(\mathcal{T}) : \widetilde{\pi}(\mathcal{T})) = mn$, as claimed. Finally, we recall that $\pi(\mathcal{T})$ is generated by the elements $\bar{c}_{ij}$ over all $i, j \in \mathbf{Z}$. Since $\bar{x}^m$ is central in $P(\mathcal{T})$, we see that $\bar{c}_{ij} = \bar{c}_{i+m,j}$, and $\bar{c}_{ij} = \bar{c}_{i,j+n}$, because $\bar{y}^n$ is also central. The last statement is then clear. □

Theorem 5.2 is an important tool for calculating tile homotopy groups. We revisit an example (3.8) we had seen earlier.

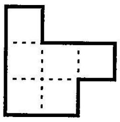
THEOREM 5.3. *The tile homotopy group of* $\mathcal{T} = \{$, $\}$ *has order* 120, *and it is a central extension of* A_5 *by* $\mathbf{Z}/2\mathbf{Z}$.

Proof. The tile path group has the presentation

$$P(\mathcal{T}) = \langle x, y \mid x^3yx^{-3}y^{-1}, xy^3x^{-1}y^{-3}, xyxyx^{-1}yx^{-1}y^{-1}x^{-1}y^{-1}xy^{-1}\rangle .$$

The relators show that $\bar{x}^3$ and $\bar{y}^3$ are central in $P(\mathcal{T})$. Let $\widetilde{P}(\mathcal{T}) = P(\mathcal{T})/\langle\bar{x}^3, \bar{y}^3\rangle = \langle x, y \mid x^3, y^3, xyxyx^{-1}yx^{-1}y^{-1}x^{-1}y^{-1}xy^{-1}\rangle$. Then the projection $P(\mathcal{T}) \twoheadrightarrow \widetilde{P}(\mathcal{T})$ induces an isomorphism of $\pi(\mathcal{T})$ onto its image $\widetilde{\pi}(\mathcal{T})$, which has index 9 in the finitely presented group $\widetilde{P}(\mathcal{T})$. Thus we can compute a presentation of $\widetilde{\pi}(\mathcal{T})$. In this particular instance, we have an even better situation, because the group $\widetilde{P}(\mathcal{T})$ turns out to be finite, and therefore $\widetilde{\pi}(\mathcal{T})$ is also finite. In fact, GAP quickly tells us that $|\widetilde{P}(\mathcal{T})| = 1080$, so that $\pi(\mathcal{T})$ has order 120, and its structure can be completely determined. □

The utility of Theorem 5.2 depends on the ability to find relations in the tile path group. It is known that this cannot be done algorithmically, but in some cases, it is easy to find the necessary relations. In Theorem 5.3, it was trivial to find them. In the next theorem, the relations are not quite as obvious.

THEOREM 5.4. *Let* $\mathcal{T} = \{$$\}$, *with all orientations allowed.*

(a) *The tile homotopy group* $\pi(\mathcal{T})$ *is solvable. Its derived series is* $\pi(\mathcal{T}) = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 = \{1\}$, *with quotients* $G_0/G_1 = \pi(\mathcal{T})^{\mathrm{ab}} = H(\mathcal{T}) \cong \mathbf{Z}\times(\mathbf{Z}/3\mathbf{Z})$, $G_1/G_2 \cong (\mathbf{Z}/2\mathbf{Z})^2$ *and* $G_2/G_3 = G_2 \cong \mathbf{Z}/2\mathbf{Z}$. *Moreover, these isomorphisms can be given explicitly.*

(b) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then (at least) one of* m *or* n *is a multiple of* 4.

(c) *A* 2×3 *rectangle has a signed tiling by* $\mathcal{T}$.

Proof. We first claim that $\bar{x}^{12}$ and $\bar{y}^{12}$ are central in $P(\mathcal{T})$. Consider the two tilings shown in Figure 5.5.

FIGURE 5.5
Two small tilings

The first shows that $\bar{x}^3$ commutes with $\bar{y}^2\bar{x}\bar{y}^2$, and the second shows that $\bar{x}^4$ commutes with $\bar{y}^2\bar{x}\bar{y}$. Therefore, $\bar{x}^{12}$ commutes with both $\bar{y}^2\bar{x}\bar{y}^2$ and $\bar{y}^2\bar{x}\bar{y}$, and thus also with $\bar{y}$. Hence $\bar{x}^{12}$ is central in $P(\mathcal{T})$, and similarly, $\bar{y}^{12}$ is also central. Let $\widetilde{P}(\mathcal{T}) = P(\mathcal{T})/\langle \bar{x}^{12}, \bar{y}^{12}\rangle$. Theorem 5.2 shows that $\pi(\mathcal{T})$ maps isomorphically onto its image in $\widetilde{P}(\mathcal{T})$, with finite index. Now we can compute a presentation of $\pi(\mathcal{T})$, using GAP. We obtain

$$G_0 = \pi(\mathcal{T}) \cong \langle z_1, z_2 \mid z_2z_1z_2z_1z_2z_1^{-2}, z_1z_2^2z_1^{-1}z_2^{-2}\rangle ,$$

where the generators are $z_1 = \bar{x}^{-1}\bar{y}\bar{x}\bar{y}^{-1}$ and $z_2 = \bar{y}^2\bar{x}\bar{y}^{-2}\bar{x}^{-1}$. From this, we find that

$$H(\mathcal{T}) = \pi(\mathcal{T})^{\mathrm{ab}} \cong \mathbf{Z} \times (\mathbf{Z}/3\mathbf{Z}) .$$

There are two different ways we can make this isomorphism explicit. Firstly, we can express the image of each c_{ij} in terms of z_1 and z_2, and then use the explicit presentation of $\pi(\mathcal{T})$ above. However, it is much easier to compute $H(\mathcal{T})$ directly. We have

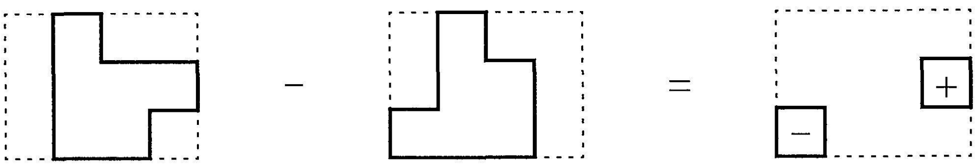


FIGURE 5.6
Translating a square 3 units to the right and 1 unit up

which shows how we can translate a square 3 units to the right and 1 unit up. By considering all 8 orientations of this relation, we find that we can translate a square by 1 diagonal unit. Now it is easy to see that $H(\mathcal{T}) \cong \mathbf{Z} \times (\mathbf{Z}/3\mathbf{Z})$ is given by $[R] \mapsto (b-r, (b+r) \bmod 3)$, where the region R contains b black squares and r red squares in the usual checkerboard coloring.

Next we compute the commutator subgroup $G_1 = [G_0, G_0]$. We cannot do this directly, because it has infinite index in G_0. However, we can utilize the same technique as in Theorem 5.2 above. The first relator implies that $z_1^3 = (z_2z_1)^3$. Therefore, z_1^3 commutes with z_2z_1, and hence is central in G_0. Now let $N = \langle z_1^3\rangle \subseteq G_0$. We see that N maps injectively to $G_0^{\mathrm{ab}} = G_0/G_1$, so that G_1 maps injectively to $G_0/N = \langle z_1, z_2 \mid z_1^3, z_2z_1z_2z_1z_2z_1^{-2}, z_1z_2^2z_1^{-1}z_2^{-2}\rangle$. Moreover, its image has index 9 in G_0/N. Now GAP can compute a presentation of G_1; it tells us that

$$G_1 \cong \langle a_1, a_2 \mid a_1^2a_2^2, a_1a_2a_1a_2^{-1}\rangle ,$$

where $a_1 = z_2 z_1 z_2^{-1} z_1^{-1}$ and $a_2 = z_2 z_1^{-1} z_2^{-1} z_1$. Also, G_1 is easily seen to be a finite group (quaternion of order 8). Thus the rest of (a) can be readily verified.

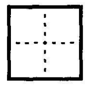
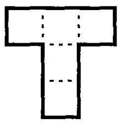
(b) It suffices to show that $\mathcal{T}$ cannot tile any $(4m+2)\times(4n+2)$ rectangle. Having already completely determined the structure of the tile homotopy group, we content ourselves with a representation proof. Define $\varphi\colon P(\mathcal{T}) \to S_{32}$ by

$$\begin{aligned}\varphi(\bar{x}) = &(1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)\\ &(21,22,23,24)(25,26,27,28)(29,30,31,32),\\ \varphi(\bar{y}) = &(1,4,32,20)(2,12,7,17)(3,24,23,11)(5,16,15,21)(6,13,27,18)\\ &(8,22,10,28)(9,19,29,25)(14,31,30,26).\end{aligned}$$

It is straightforward to check that this indeed gives a homomorphism; one only needs to verify that the boundary words of all eight orientations are in the kernel of φ. We also note that $\varphi(\bar{x}^{4m+2}\bar{y}^{4n+2}\bar{x}^{-(4m+2)}\bar{y}^{-(4n+2)})$ is non-trivial, so a $(4m+2)\times(4n+2)$ rectangle cannot be tiled by $\mathcal{T}$.

(c) This follows from the explicit isomorphism $H(\mathcal{T}) \cong \mathbf{Z}\times(\mathbf{Z}/3\mathbf{Z})$ given above. Also, an explicit signed tiling is easy to give, based upon Figure 5.6 above. □

We remark that these computations depend upon the correctness of the computer program. If a proof of non-tileability relies on this computation, it may be advantageous to give a *certificate of proof*, namely a homomorphism $P(\mathcal{T}) \to G$ to a group in which we can compute easily. Having done that, the representation proof can be easily verified, and is less susceptible to error.

THEOREM 5.7. *Let* $\mathcal{T} = \{$ ,  $\}$, *with all orientations allowed.*

(a) *The tile homotopy group of* $\mathcal{T}$ *has order* 32 *and is a central extension of* $(\mathbf{Z}/2\mathbf{Z})^4$ *by* $\mathbf{Z}/2\mathbf{Z}$.

(b) *The tile homology group,* $H(\mathcal{T}) \cong (\mathbf{Z}/2\mathbf{Z})^4$, *and a specific isomorphism is given as follows. Suppose that the region* R *covers* X_0 *[respectively,* X_1, X_2*] cells with* x*-coordinate congruent to* $0 \bmod 3$ *[respectively,* $1 \bmod 3$, $2 \bmod 3$*]. Also, suppose that* R *covers* Y_0 *[respectively,* Y_1, Y_2*] cells with* y*-coordinate* $\equiv 0 \bmod 3$ *[respectively,* $1 \bmod 3$, $2 \bmod 3$*]. Then a specific isomorphism* $H(\mathcal{T}) \cong (\mathbf{Z}/2\mathbf{Z})^4$ *is given by*

$$[R] \mapsto ((X_0+X_1) \bmod 2, (X_1+X_2) \bmod 2, (Y_0+Y_1) \bmod 2, (Y_1+Y_2) \bmod 2).$$

(c) *If* $\mathcal{T}$ *tiles an* $m\times n$ *rectangle, then* mn *is even.*

(d) *A* 3×3 *square has a signed tiling by* $\mathcal{T}$.

Proof. (a) We first claim that $\bar{x}^6$ is central in $P(\mathcal{T})$. Consider the two tilings below.

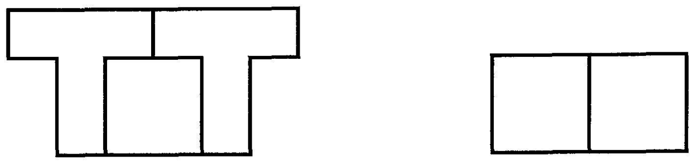


FIGURE 5.8
Two small tilings

They show that

$$\bar{y}^{-2}\bar{x}^4\bar{y}^2\bar{x}\bar{y}\bar{x}^{-6}\bar{y}^{-1}\bar{x} = 1 \quad \text{and} \quad \bar{y}^{-2}\bar{x}^4\bar{y}^2\bar{x}^{-4} = 1\,,$$

so that $\bar{x}\bar{y}\bar{x}^{-6}\bar{y}^{-1}\bar{x} = \bar{x}^{-4}$. This shows that $\bar{x}^6$ commutes with $\bar{y}$ and therefore is central. Similarly, $\bar{y}^6$ is central in $P(\mathcal{T})$. Now let $\widetilde{P}(\mathcal{T}) = P(\mathcal{T})/\langle \bar{x}^6, \bar{y}^6\rangle$. Theorem 5.2 shows that $\pi(\mathcal{T})$ maps isomorphically onto its image in $\widetilde{P}(\mathcal{T})$, and it has index 36. We can now compute

$$\pi(\mathcal{T}) \cong \langle z_1, z_2, z_3, z_4 \mid z_1^2, z_2^2, z_3^2, z_4^4, (z_1z_2)^2z_4^2, (z_1z_3)^2z_4^2, (z_2z_3)^2z_4^2, (z_1z_4)^2, (z_2z_4)^2, (z_3z_4)^2\rangle$$

where $z_1 = \bar{y}\bar{x}\bar{y}^{-1}\bar{x}^{-1}$, $z_2 = \bar{y}\bar{x}^{-1}\bar{y}^{-1}\bar{x}$, $z_3 = \bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}^{-2}$ and $z_4 = \bar{y}^2\bar{x}\bar{y}^{-2}\bar{x}^{-1}$. We can easily check that this group is finite, and its structure can be completely determined. In fact, the relators make it clear that z_4^2 is central, has order 2, generates the commutator subgroup, and the quotient $\pi(\mathcal{T})/\langle z_4^2\rangle$ is an elementary abelian 2-group of rank 4.

(b) We show how we can translate a square by 3 units.

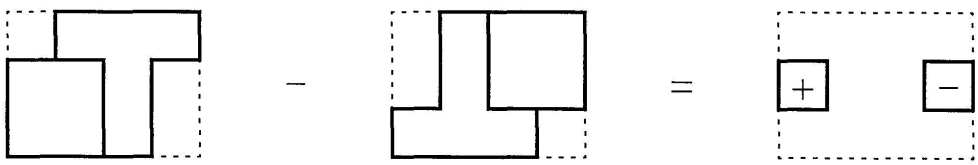


FIGURE 5.9
Translating a square by 3 units

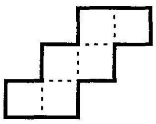
Now a straightforward computation, similar to Examples 2.5 and 2.7, shows that $H(\mathcal{T}) \cong (\mathbf{Z}/2\mathbf{Z})^4$, and the isomorphism is as claimed.

(c) We must show that $\mathcal{T}$ cannot tile a $(2m+1)\times(2n+1)$ rectangle, so it suffices to show that $\mathcal{T}$ cannot tile a $(6m+3)\times(6n+3)$ rectangle. We use a representation proof. Define a homomorphism $\varphi\colon P(\mathcal{T}) \to S_{48}$ by

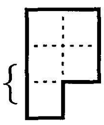
$$\begin{aligned}\varphi(\bar{x}) =&(1,13,11,12,10,16)(2,41,34,25,38,31)(3,42,35,26,39,32)(4,40,36,27,37,33)\\ &(5,20,46,6,23,43)(7,19,17,48,22,14)(8,28,9,29,45,30)(15,44,21,18,47,24),\\ \varphi(\bar{y}) =&(1,27,30,12,10,23,29,18,11,8,28,31)(2,13,14,46,4,44,43,45,3,25,15,40)\\ &(5,20,24,35,9,21,41,34,33,19,16,36)(6,26,39,22,47,42,38,17,48,32,37,7).\end{aligned}$$

It is straightforward to verify that this indeed defines a homomorphism. Furthermore, we easily check that $\varphi(\bar{x}^{6m+3}\bar{y}^{6n+3}\bar{x}^{-(6m+3)}\bar{y}^{-(6n+3)})$ is non-trivial, so a $(6m+3)\times(6n+3)$ rectangle cannot be tiled by $\mathcal{T}$.

(d) This follows from the isomorphism $H(\mathcal{T}) \cong (\mathbf{Z}/2\mathbf{Z})^4$ given in part (b). Also, it is easy to give an explicit one, based upon Figure 5.9. □

REMARK 5.10. The tilings in Figure 5.8 and the argument involved essentially amount to "untiling" two square tetrominoes from the left figure. This is the non-abelian analogue of a signed tiling. Since the boundary word of the 1×6 rectangle is trivial in $P(\mathcal{T})$, Theorem 5.7 remains true even if this rectangle is included in the protoset $\mathcal{T}$. We can also show that the hexomino  has such a "generalized tiling" by $\mathcal{T}$, so this shape may also be included in $\mathcal{T}$, and Theorem 5.7 remains valid.

We give one more example.

THEOREM 5.11. *Let* $\mathcal{T} = \{$$\}$, *where rotations are allowed, but reflections are prohibited.*

(a) *The tile homotopy group, $\pi(\mathcal{T})$, is a central extension of $\mathbf{Z}^4$ by $\mathbf{Z}/2\mathbf{Z}$. In particular, it is solvable.*

(b) *The tile homology group is $H(\mathcal{T}) \cong \mathbf{Z}^4$, and an explicit isomorphism is given as follows. Suppose that the region R covers n_0 [respectively, n_1, n_2, n_3, n_4] cells (i,j) with $2i+j \equiv 0 \bmod 5$ [respectively, $1 \bmod 5$, $2 \bmod 5$, $3 \bmod 5$, $4 \bmod 5$]. Then an explicit isomorphism $H(\mathcal{T}) \xrightarrow{\cong} \mathbf{Z}^4$ is given by $[R] \mapsto (n_1-n_0, n_2-n_0, n_3-n_0, n_4-n_0)$.*

(c) *If $\mathcal{T}$ tiles an $m\times n$ rectangle, then mn is even.*

(d) *A 1×5 rectangle has a signed tiling by $\mathcal{T}$.*

Proof. (a) Note that $\mathcal{T}$ tiles a 2×5 rectangle, which implies that $\bar{x}^2$ commutes with $\bar{y}^5$. Similarly, $\bar{x}^5$ commutes with $\bar{y}^2$. Therefore, $\bar{x}^{10}$ commutes with $\bar{y}$ and thus is central in $P(\mathcal{T})$. In the same way, $\bar{y}^{10}$ is also central in $P(\mathcal{T})$, so we can compute a presentation of $\pi(\mathcal{T})$, using Theorem 5.2. We obtain a presentation for $\pi(\mathcal{T})$ with 5 generators: $z_1 = \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$, $z_2 = \bar{y}\bar{x}^{-1}\bar{y}^{-1}\bar{x}$, $z_3 = \bar{x}^{-1}\bar{y}^{-1}\bar{x}\bar{y}$, $z_4 = \bar{y}^{-1}\bar{x}\bar{y}\bar{x}^{-1}$ and $w = z_1 z_2 z_1^{-1} z_2^{-1}$. The relations are $w^2 = 1$, $wz_i = z_i w$ for $1 \le i \le 4$, and $z_i z_j z_i^{-1} z_j^{-1} = w$ for $1 \le i < j \le 4$. The relations show that w is central in $\pi(\mathcal{T})$ and that the quotient $\pi(\mathcal{T})/\langle w \rangle$ is isomorphic to $\mathbf{Z}^4$. Furthermore, w has order 2, and it generates the commutator subgroup of $\pi(\mathcal{T})$. This proves (a).

(b) Note that we have

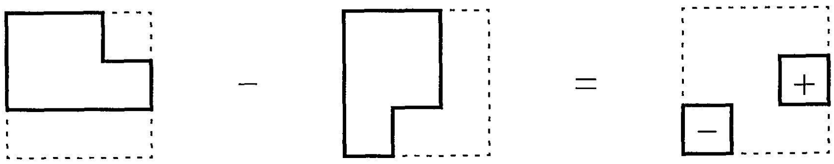


FIGURE 5.12
Translating a square 2 units to the right and 1 unit up

so that $\bar{a}_{ij} = \bar{a}_{i+2,j+1}$ in $H(\mathcal{T})$. Similarly, we have $\bar{a}_{ij} = \bar{a}_{i-1,j+2}$, so $H(\mathcal{T})$ is generated by $\bar{a}_{00}, \bar{a}_{10}, \bar{a}_{20}, \bar{a}_{30}$ and $\bar{a}_{40}$. Furthermore, the relations collapse into a single relation: $\bar{a}_{00} + \bar{a}_{10} + \bar{a}_{20} + \bar{a}_{30} + \bar{a}_{40} = 0$. Thus $H(\mathcal{T}) \cong \mathbf{Z}^4$, and the isomorphism is as claimed.

(c) It suffices to show that $\mathcal{T}$ cannot tile a $(10m+5) \times (10n+5)$ rectangle. We use a representation proof. Define a homomorphism $\varphi\colon P(\mathcal{T}) \to S_{64}$ by

$$\begin{aligned}\varphi(\bar{x}) =&(1,2,4,47,16,27,41,54,56,9)(3,6,12,11,34,50,62,61,49,58)\\ &(5,10,19,32,24,36,31,37,42,55)(7,14,23,28,43,57,52,40,38,46)(8,59)\\ &(13,21,35,51,20,15,25,17,18,30)(22,33,48,60,64,26,39,53,63,44)(29,45),\\ \varphi(\bar{y}) =&(2,3,5,9,17,28,42,12,14,22)(4,7,13,6,20)(8,25,37,11,33)\\ &(10,18,29,44,58)(15,24,30,46,57,63,62,48,54,47)(16,26,38,50,61)\\ &(19,31,39,45,21,34,49,51,59,64)(27,40)(32,36,52,35,41)(43,56,60,55,53).\end{aligned}$$

As usual, it is straightforward to verify that φ indeed defines a homomorphism, and that $\varphi(\bar{x}^{10m+5}\bar{y}^{10n+5}\bar{x}^{-(10m+5)}\bar{y}^{-(10n+5)})$ is non-trivial.

(d) This follows from the explicit isomorphism $H(\mathcal{T}) \xrightarrow{\cong} \mathbf{Z}^4$ given above. Alternatively, it is easy to give a signed tiling, based upon Figure 5.12. $\square$