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interpreting the former as elementary symmetric functions in certain roots of $G$ (or their squares).

Ehresmann [E] investigated the topology of complex Grassmann manifolds (and other hermitian symmetric spaces) by studying the algebra of $K$-invariant differential forms on them ($K = U(N)$ for $X = G(m,n)$). This relies on the fact that the invariant forms are harmonic for the natural hermitian structure on $X$, which implies that the ring of all such forms is isomorphic to $H^*(X)$. Kostant [K1] [K2] later found analogues of these results for arbitrary (generalized) flag manifolds. The representation theory used to determine the $K$-invariant forms in this program does not directly relate the multiplicities $c_{\lambda\mu}^{\nu}$ in equations (1) and (2). Note however that the cited works of É. Cartan and Ehresmann were used by Chern in his fundamental paper on the characteristic classes of complex manifolds [Ch1]. More recently, Stoll [St] used fiber integration to study the algebra of invariant forms on the Grassmannian, but his work does not address the question posed in the Introduction.

Following [SGA6], [V] and [Be], the isomorphism between $\operatorname{gr} \mathrm{R}(G)$ and $\mathrm{Pol}(\mathfrak{g})^G$ in §6 may be used to construct the Chern-Weil (or characteristic) homomorphism in algebraic geometry. Let $P \to X$ be a principal $G$-bundle over a smooth algebraic variety $X$ and let $CH^*(X)$ denote the Chow group of algebraic cycles on $X$ modulo rational equivalence. The Grothendieck group $K(X)$ of vector bundles on $X$ is a $\lambda$-ring, with the $\lambda$-operations induced by exterior powers. According to [SGA6, Exp. XIV], the graded ring $\operatorname{gr} K(X) \otimes \mathbf{R}$ is canonically isomorphic to the real Chow ring $CH^*_{\mathbf{R}}(X) = CH^*(X) \otimes \mathbf{R}$. There is a natural $\lambda$-ring homomorphism $\pi \colon \mathcal{R}(G) \to K(X)$, defined by sending a representation $G \to \mathrm{GL}(E)$ to the associated vector bundle $P \times_G E$ over $X$. The characteristic homomorphism is the induced map

$$\operatorname{gr}(\pi)_{\mathbf{R}} \colon \mathrm{Pol}(\mathfrak{g})^G \longrightarrow CH^*_{\mathbf{R}}(X) .$$

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