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HYPERBOLIC POLYGONS AND SIMPLE CLOSED GEODESICS

by Hugo PARLIER *)

ABSTRACT. On a closed hyperbolic Riemann surface S of genus $g > 1$, we study the existence of a universal constant related to simple closed geodesics. For a simple closed geodesic γ on S, we give elementary proofs of the following facts: there is always a simply connected disk of radius $r_\gamma > \frac{1}{2}\ln 3$ imbedded in $S \setminus \gamma$, and conversely, for *any* surface S, the infimum of the values r_γ (for all simple closed geodesics γ on S) is *always* equal to $\frac{1}{2}\ln 3$. The proofs are based on the relationship between right-angled hyperbolic polygons and simple closed geodesics.

1. INTRODUCTION

The main goal of this article is to give an elementary proof of the following fact:

THEOREM 1.1. *Let S be a closed Riemann surface of genus $g > 1$, endowed with a metric of constant curvature -1. Let γ be a simple closed geodesic on S. The set $S \setminus \gamma$ contains $4g-4$ closed disks of radius $\frac{1}{2}\ln 3$. Conversely, if $\rho > \frac{1}{2}\ln 3$ is a given constant, there exists a simple closed geodesic γ_ρ on S such that the set $S \setminus \gamma_\rho$ does not contain any open disks of radius ρ.*

For specialists in the area, this can be deduced from well-known results concerning laminations, and in particular maximal laminations, and their relationship to simple closed geodesics. The goal here is to give a step by step

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proof. The different steps have an interest in their own right, and the process allows us to get a feeling for some of the fundamental aspects of negatively curved surfaces. Through examples and a careful study of hyperbolic polygons, one can hope to get a feeling for why this theorem is true, and subsequently some insight into the nature of laminations.

The standpoint is close in nature to that of other articles concerning the size of embedded disks on hyperbolic Riemann surfaces, namely an article by Yamada [10] and another by Bavard [1]. In the first article, a sharp lower bound on the maximal injectivity radius of a compact hyperbolic surface is given. This gives an exact value for the Margulis constant in dimension 2, sometimes called Marden's universal constant for Fuchsian groups [9]. In the second article, for any given genus g, a sharp upper bound on the maximal injectivity radius of a surface of genus g is obtained. However, in both articles, the sharp bounds are reached by *specific surfaces*. In contrast, the surprising part of the result presented here is the universality of the constant $\frac{1}{2}\ln 3$, as it depends neither on the genus nor on a specific choice of surface.

The remainder of the paper is divided into five sections. After a standard section dedicated to definitions, notations and known results, the first subject treated is a detailed study of hyperbolic right-angled polygons. Following this, the existence of a disk of radius $\frac{1}{2}\ln 3$ on the complement of a geodesic is proved. The penultimate section deals with the optimality of the constant, firstly by constructing specific examples of surfaces and simple closed geodesics, and secondly by a construction of a simple closed geodesic on an arbitrary surface. The final section is about the relationship of the constant with Marden's universal constant for Fuchsian groups.

2. PRELIMINARIES

Here a *surface* will generally be a compact Riemann surface equipped with a metric of constant curvature -1, with or without boundary. Such a surface is always locally isometric to the hyperbolic plane $\mathbf{H}$. A surface will generally be represented by S, and distance on S (between points, curves or other subsets) by $d_S(\cdot,\cdot)$. If the surface is closed (without boundary) then its genus will generally be g and otherwise its signature will be (g,n), where n designates the number of boundary curves. All boundary curves are assumed to be smooth simple closed geodesics, although much of what is said about surfaces with boundary also holds for surfaces with cusps.

The *Euler characteristic* for a surface S of signature (g,n) is $\chi(S) = 2-2g-n$. Notice that $\chi(S)<0$ in our case. A surface of signature $(0,3)$ is called a Y*-piece* or a *pair of pants* and will generally be represented by $\mathcal{Y}$ or $\mathcal{Y}_k$. A surface of signature $(0,4)$ is sometimes referred to as an X*-piece*. The Teichmüller space for closed surfaces of genus g is denoted by $\mathcal{T}_g$. For this article, the Teichmüller space can be thought of as the space parametrized by the *Fenchel-Nielsen* length and twist parameters (see for instance [4]).

A curve, unless specifically mentioned, will always be non-oriented. A closed curve will be considered *primitive*, meaning that it cannot be written as the k-fold iterate of another closed curve. A *non-trivial curve* on S is a curve which is not freely homotopic to a point. A closed curve on S is called *simple* if it has no self-intersections. Closed curves (geodesic or not) will generally be represented by Greek letters (α, β, γ and γ_i etc.) whereas paths (geodesic or not) will generally be represented by lower-case letters (a, b etc). The geometric intersection number between two distinct curves α and β will be denoted $\text{int}(\alpha,\beta)$. Unless otherwise specified, a *geodesic* is a simple closed geodesic curve. The set of all simple closed geodesics on S will be denoted by $\mathbf{G}(S)$. A *non-separating closed curve* is a closed curve γ such that the set $S\setminus\gamma$ is connected. Otherwise a closed curve is called *separating*. A set of disjoint simple closed geodesics that decompose the surface into Y-pieces is called a *partition* and shall generally be denoted by $\mathcal{P}$. (A partition is often called the set of pants decomposition geodesics.) The function that associates to a finite path or curve its length will be represented by $\ell(\cdot)$, although generally a path or a curve's name and its length will not be distinguished.

We will readily make use of the following fact. Let S be a surface with a given partition $\mathcal{P}$ and suppose that $\gamma\in\mathcal{P}$. The geodesic γ is the boundary of either one (case 1) or two (case 2) distinct Y-pieces in $\mathcal{P}$. In case 1, γ is the interior geodesic of a surface of signature $(1,1)$. In case 2, γ is the interior geodesic of a surface of signature $(0,4)$. Let $\delta\in\mathbf{G}(S)$ such that γ is the only geodesic in $\mathcal{P}$ that intersects δ. Furthermore, δ can be chosen such that $\text{int}(\gamma,\delta)=1$ in case 1 and $\text{int}(\gamma,\delta)=2$ in case 2. Let $k\in\mathbf{Z}$ and let the result of k Dehn twists around δ on γ be denoted by $\mathcal{D}_{k,\delta}(\gamma)$. Notice that for any k, $\mathcal{P}'=\{\mathcal{P}\setminus\gamma\}\cup\mathcal{D}_{k,\delta}(\gamma)$ is a partition on S. The convexity of geodesic length functions along earthquake paths [8] implies that k can be so chosen that $\ell(\mathcal{D}_{k,\delta}(\gamma))$ is arbitrarily large.

DEFINITIONS. Let S be a compact surface. Let γ be a simple closed geodesic on S and $x_0 \in S$. We define:

(1) $D(x_0, r) = \{x \in S \mid d_S(x_0, x) < r\}$. This should be seen as a distance set (i.e., it is not necessarily an embedded hyperbolic disk).

(2) $r_{S,\gamma,x} = \sup\{r \mid D(x,r)$ is homeomorphic to a disk and $D(x,r) \cap \gamma = \varnothing\}$. This is the maximal radius for an *open* disk centered in x that does not intersect γ.

(3) $r_{S,\gamma} = \sup\{r_{S,\gamma,x} \mid x \in S\}$. For a given geodesic γ, this is the maximal radius for an open disk on S that does not intersect γ.

(4) $r_S = \inf\{r_{S,\gamma} \mid \gamma \in \mathbf{G}(S)\}$.

(5) $r_g = \inf\{r_S \mid S \in \mathcal{T}_g\}$.

As a prelude we consider the sphere $\mathbf{S}^2$ and the torus $\mathbf{T}^2$, equipped with their standard metrics of constant curvature 1 and 0, and determine the above values for these surfaces. On $\mathbf{S}^2$ the (simple) closed geodesics are great circles. It is easy to see that $r_{\mathbf{S}^2} = \pi/2$ and thus $r_0 = \pi/2$. For $\mathbf{T}^2$ the situation is even more radical. Here $r_{\mathbf{T}^2} = r_1 = 0$. This is of course because for a given $\epsilon > 0$, there exists a geodesic γ on $\mathbf{T}^2$ such that there are no embedded disks of radius ϵ on $\mathbf{T}^2$ that do not intersect γ.

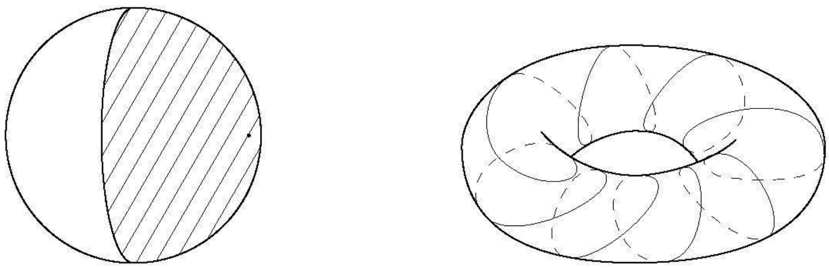

FIGURE 1
The $\mathbf{S}^2$ and $\mathbf{T}^2$ cases

In the hyperbolic case with S a surface of genus $g \geq 2$, the result of Birman and Series on the non-density of simple closed geodesics [3] implies that for a given S with $g \geq 2$, $r_S > 0$, but this does not give any information on r_g. Using these notations, the main result of the paper gives explicit values to both r_S and r_g.

3. Properties of right-angled polygons

The decomposition of Y-pieces into two isometric right-angled hyperbolic hexagons encourages a detailed study of hyperbolic polygons. Unless otherwise specified, an n-gon will always be considered to be right-angled and hyperbolic. We will use the following well-known propositions (for proofs see [2], [4], or [7]):

PROPOSITION 3.1. *Let P be a pentagon with adjacent edges a and b. Let c be the only remaining edge adjacent neither to a nor to b. Then*

$$\sinh a \sinh b = \cosh c .$$

PROPOSITION 3.2. *Let H be a hexagon with a, b and c non-adjacent edges. Let $\tilde{c}$ be the edge adjacent to a and b. Then*

$$\cosh c = \sinh a \sinh b \cosh \tilde{c} - \cosh a \cosh b .$$

From Proposition 3.1 the following is easily deducted.

LEMMA 3.3. *On any pentagon P there exists a pair of adjacent edges a, b with* $\sinh a > 1$, $\sinh b > 1$.

The idea is now to place a disk tangent to two adjacent edges of lengths a,b with $\sinh a > 1$ and $\sinh b > 1$.

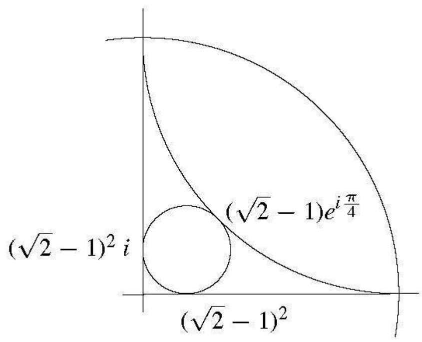

FIGURE 2
The hyperbolic triangle with vertices 0, 1 and i

Let T be the generalized triangle with vertices 0, 1, i in the unit disk model of $\mathbf{H}$. The three edges of the triangle are the segment $[0,1[$, the segment $[0,i[$ and the segment $]i,1[$ as in Figure 2. The disk tangent to all three segments is well-defined and the tangent points are $(\sqrt{2}-1)^2$, $i(\sqrt{2}-1)^2$ and $(\sqrt{2}-1)e^{i\frac{\pi}{4}}$. One can deduce the Euclidean radius of the disk as $(\sqrt{2}-1)^2$, and the hyperbolic radius $\frac{1}{2}\ln(\frac{9+4\sqrt{2}}{7})$ which we shall denote by ρ_p.

PROPOSITION 3.4. *Let P be a pentagon and $\overset{\circ}{P}$ its interior. $\overset{\circ}{P}$ contains a closed disk of radius ρ_p. Conversely, for $\rho > \rho_p$ there exists a pentagon Q such that no open disk of radius ρ is contained in $\overset{\circ}{Q}$.*

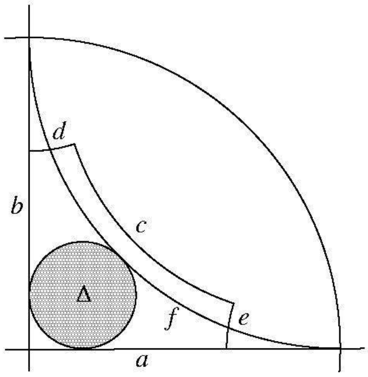

FIGURE 3
A hyperbolic pentagon

Proof. By Lemma 3.3, P always has two adjacent edges a, b such that $\sinh a > 1$ and $\sinh b > 1$. Place P such that the intersection of a and b is at the origin, and such that a is on the positive real axis, and b is on the positive imaginary axis. The remaining edges shall be denoted c, d and e as in Figure 3. Place a disk Δ centered on the $x = y$ axis tangent to a and b of radius ρ_p. The disk does not touch the edges c and e because both $a, b > \operatorname{arcsinh} 1$. Let f denote the hyperbolic line which passes through i and 1. By comparing edge c with f it is easy to see that c is further away from the origin than f. Because the edge of the disk touches only a and b one can easily slide the center along the $x = y$ axis to obtain a closed disk contained in $\overset{\circ}{P}$. To show the sharpness of the constant ρ_p we shall construct a pentagon which contains no disk of radius $\rho > \rho_p$. Let $\rho - \rho_p = \epsilon$. Let p be the point on the $x = y$ axis at a distance $\epsilon/4$ further away from 0 than $(\sqrt{2}-1)e^{i\frac{\pi}{4}}$, as in Figure 4.

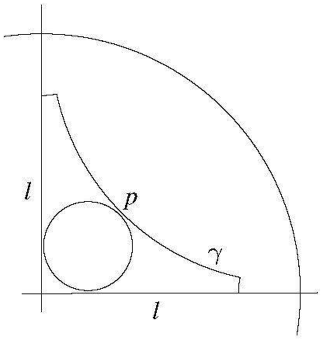

FIGURE 4
The l-pentagon

The geodesic axis $x = y$ admits a unique geodesic γ perpendicular at p. There is a unique pentagon with edges on both the real and imaginary axes and on γ. This pentagon does not contain an open disk of radius ρ in its interior. □

REMARK 3.5. The example constructed in the proof will be called the l-pentagon. It has two adjacent edges with hyperbolic length l-arbitrarily long and equal, and thus a third edge which is defined by Proposition 3.1. The two remaining edges have lengths that decrease towards 0 as l increases. These edges will be referred to as the short edges. This pentagon has one degree of freedom $l > \operatorname{arcsinh} 1$ being the equal length of the first two adjacent edges.

The following proposition shows that the example of l-pentagons is not isolated.

PROPOSITION 3.6. *Let P be a pentagon with edges labelled as in Figure 3. For $\epsilon > 0$, there exists x_ϵ such that for $a, b > x_\epsilon$, we have $d, e < \epsilon$.*

Proof. The values a, b and c verify

$$\sinh a \sinh b = \cosh c .$$

Also d, c and a verify

$$\sinh c \sinh d = \cosh a .$$

Notice that for $\theta \in \mathbf{R}$, $\sinh \theta \geq \cosh \theta - 1$. Thus:

$$\sinh d = \frac{\cosh a}{\sinh c} \leq \frac{\cosh a}{\cosh c - 1} = \frac{\cosh a}{\sinh a \sinh b - 1} .$$

For a given ϵ, a and b can thus be chosen sufficiently large so that $\sinh d$ is less than $\sinh \epsilon$. This gives us a condition $a, b > x_d$. The same procedure applied to e gives a condition $a, b > x_e$. We can now choose $x_\epsilon = \max\{x_d, x_e\}$. □

COROLLARY 3.7. *For $\epsilon > 0$ there exists x_ϵ so that the following is true. All pentagons (with edges labeled as in Figure 3) with $a, b > x_\epsilon$ contain no disks of radius $\rho_p + \epsilon$ in their interior.* □

All n-gons with $n \geq 5$ can be cut along perpendicular geodesic lines to certain edges to obtain a pasting of $n-4$ pentagons. As a consequence $n-4$ disjoint closed disks of radius ρ_p are always contained in the interior of an n-gon. This constant remains sharp for the n-gon, and this is proved by the following example: take $n-4$ isometric l-pentagons and paste them along the short edges (Figure 5).

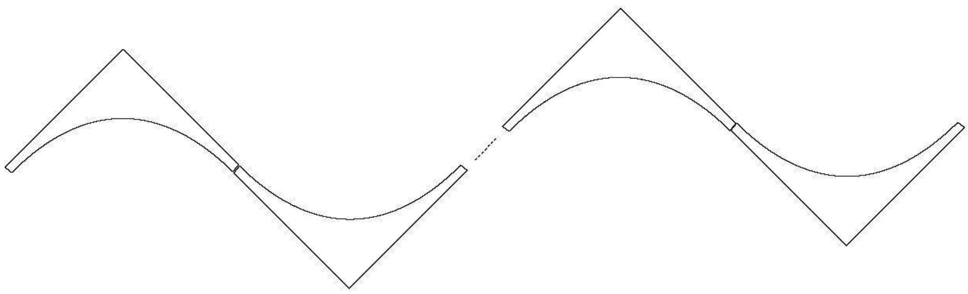

FIGURE 5
$n-4$ l-pentagons pasted along short edges

The result obtained is an n-gon. Let $\rho > \rho_p$ be a constant. l can be increased so that none of the $n-4$ pentagons contain an open disk of radius ρ. Since the pasting was done along the short edges, it is not possible to put a disk centered on the common border between two pentagons. Thus there exists an n-gon which does not contain any disk of radius ρ in its interior. This proves the following:

PROPOSITION 3.8. *All n-gons, $n \geq 5$, contain $n-4$ disjoint closed disks of radius ρ_p. Conversely, for $\rho > \rho_p$, there exists an n-gon which does not contain a disk of radius ρ in its interior.* □

4. THE DISK OF RADIUS $\frac{1}{2}\ln 3$

This section leads to the first main step towards answering the initial question. It will be shown that on a closed Riemann surface S of genus $g \geq 2$, for a given simple closed geodesic γ, there is always a disk of radius $\rho_s = \frac{1}{2}\ln 3$ left untouched by γ.

A given simple closed geodesic γ can always be completed into a partition that decomposes S into Y-pieces. A Y-piece can always be decomposed into two isometric hexagons, and then further into four pentagons, each pentagon isometric to at least one other. Proposition 3.4 proves that there is always a closed disk of radius ρ_p on S that leaves γ untouched. The symmetry of a Y-piece allows for even larger disks.

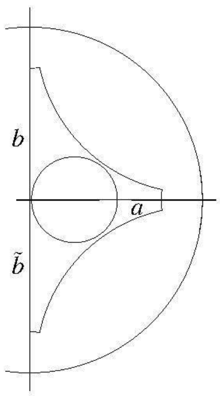

FIGURE 6
The disk of radius ρ_s

The constant $\rho_s = \frac{1}{2}\ln 3$ is obtained on a hexagon as in Figure 6. The hexagon is obtained by gluing two pentagons along edges of equal length. The pasting is along the real axis in Figure 6. Of course, ρ_s is the radius of the maximally embedded open disk in an ideal hyperbolic triangle.

LEMMA 4.1. *In order to insert a closed disk of radius ρ_s (centered on the real axis) into the interior of the hexagon, the following conditions are sufficient:*

(1) $a > \ln 3$,

(2) $b, \tilde{b} > \ln \frac{1+\sqrt{5}}{2}$.

Proof. The first condition is obvious, and the second is the result of a straightforward calculation. □

In fact, the conditions of the above lemma are necessary as well, if one is concerned about where the center of the disk should be.

As mentioned before, we will frequently make use of properties of right-angled polygons and, in particular, that consecutive edges of pentagons verify the equality of Proposition 3.1. This implies the following result (for any right-angled polygon although we will use it only for pentagons and hexagons).

LEMMA 4.2. *If r and s are consecutive edges of a right-angled polygon, then*

$$r \leq \ln 3 \quad \Longrightarrow \quad s > \frac{1}{2}\ln 3 \ \left(> \ln \frac{1+\sqrt{5}}{2} \right),$$

$$r \leq \frac{1}{2}\ln 3 \quad \Longrightarrow \quad s > \ln 3 . \quad \square$$

There are other sufficient conditions for the embedding of a disk of radius ρ_s which we shall also use. These make use of geodesic paths which we shall call *heights*, which are the three paths that separate a hexagon into two right-angled pentagons. These three paths join opposite edges of a hexagon. For example, path a in Figure 6 is a height. These paths share properties with the usual notion of heights of a triangle, such as the fact that they intersect in a single point.

LEMMA 4.3. *Let H be a hexagon with two consecutive edges a and b that verify $a, b > \ln 3$ and suppose that the two heights h_a and h_b that intersect a and b also verify $h_a, h_b > \ln 3$. Then H has an embedded closed disk of radius ρ_s.*

Proof. Consider a disk of radius ρ_s, centered on the bisector of a and b, tangent to both a and b. If this disk touches any of the remaining edges then at least one of the lengths a, b, h_a or h_b is of length less or equal to $\ln 3$. $\square$

LEMMA 4.4. *Let H be a hexagon whose heights h_k ($k \in \{1,2,3\}$) satisfy $h_k > \ln 3$. Then H has an embedded closed disk of radius ρ_s.*

Proof. By the previous lemma, if H has two consecutive edges of length superior to $\ln 3$, then H contains such a disk. Denote by a, b and c three non-adjacent edges of H, and by $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ the three edges diametrically opposite. Thus we may suppose that $a < \ln 3$. One of the two arcs of a

separated by a height is of length less than $\frac{1}{2}\ln 3$, which implies that one of the edges adjacent to a has length strictly greater than $\ln 3$ (by Lemma 4.2). Let us suppose that this edge is $\tilde{b}$. The other edge adjacent to $\tilde{b}$ is c. Now if $c > \ln 3$ the previous lemma guarantees the existence of the embedded disk. If however $c \leq \ln 3$, consider the two arcs of $\tilde{b}$, say $\tilde{b}_1$ and $\tilde{b}_2$. By Lemma 4.2, both $\tilde{b}_1$ and $\tilde{b}_2$ are of length strictly greater than $\frac{1}{2}\ln 3$. By Lemma 4.1, there is thus a disk on the height leaving from $\tilde{b}$. □

As seen in the previous section, not all hexagons have embedded disks of radius ρ_s, so this condition on heights is not always true. However, the following is.

LEMMA 4.5. *Let H be a hexagon with a height $h_1 < \ln 3$. Then the two remaining heights h_2 and h_3 verify $h_2, h_3 > \ln 3$.*

Proof. Notice that each height separates the hexagon into pentagons. Suppose that two heights, say h_1 and h_2, both verify $h_1, h_2 < \ln 3$. Consider the two pentagons P_1 and P_2 that are separated by h_1. One arc of h_2, say h_{21}, is found on P_1 and the other one h_{22} is found on P_2. One of them, say h_{21}, has length less than $\frac{1}{2}\ln 3$. This implies that one of the edges of P_1 adjacent to h_1 must be of length less than $\frac{1}{2}\ln 3$, which is impossible by Lemma 4.2. □

We are now well equipped to prove the following

PROPOSITION 4.6. *Let $\mathcal{Y}$ be a Y-piece and α, β, γ its three geodesic boundary curves. There are always two closed disks of radius ρ_s on $\mathcal{Y} \setminus (\alpha \cup \beta \cup \gamma)$. For $\rho > \rho_s$ there exists a Y-piece that does not contain an open disk of radius ρ in its interior.*

Proof. Consider the decomposition of $\mathcal{Y}$ into two isometric hexagons H and $\widetilde{H}$ obtained by cutting along simple geodesic perpendicular paths between the boundary curves. Denote by a (resp. b, c) the perpendicular path between β and γ (resp. between α and γ, between α and β).

If the heights of these hexagons verify the conditions of Lemma 4.4, then both H and $\widetilde{H}$ contain an embedded disk of radius ρ_s, and the result is verified. Thus we must suppose that a height of H, say the height h_c between c and γ, has length $h_c \leq \ln 3$. It follows from Lemma 4.2 that $c > \ln 3$.

From Lemma 4.5, it follows that the two remaining heights of H, say h_a and h_b, satisfy h_a, $h_b > \ln 3$. (The notations h_a and h_b are as in Figure 7. The three corresponding heights for $\tilde{H}$ will be denoted by $\tilde{h}_a$, etc.)

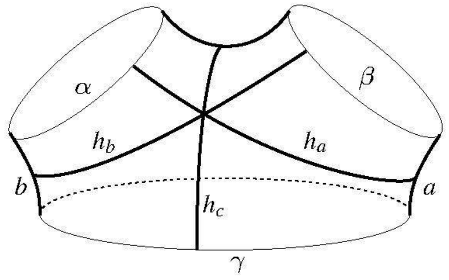

FIGURE 7
$\mathcal{Y}$ and certain paths

Denote the paths γ_{ij} as in Figure 8. Because of the symmetry of the Y-piece, $\gamma_{11} = \gamma_{21}$ and $\gamma_{12} = \gamma_{22}$. Again, because $h_c \leq \ln 3$, Lemma 4.2 implies that $\gamma_{ij} > \frac{1}{2} \ln 3$ for all $i, j \in \{1, 2\}$.

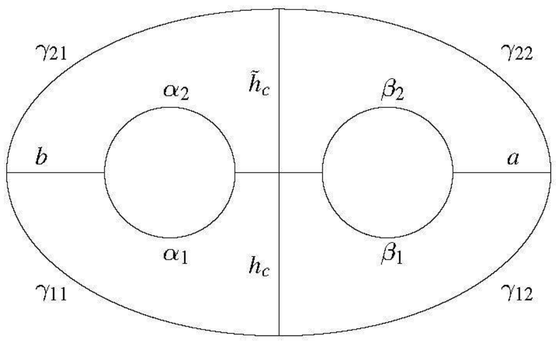

FIGURE 8
A pair of pants viewed by a topologist

Now suppose $a, b > \ln 3$. Lemma 4.1 now implies that there are two disjoint closed disks on $\mathcal{Y}$, one centered on path a and the other on path b.

Suppose that $a > \ln 3$ and $b \leq \ln 3$. By cutting $\mathcal{Y}$ along paths $h_b \cup \tilde{h}_b$ and b one obtains two (in general non-isometric) hexagons. Consider the notations of Figure 8. The paths α_1 and α_2 verify $\alpha_1, \alpha_2 > \frac{1}{2} \ln 3$. By Lemma 4.1, the hexagon containing c contains a closed disk of radius ρ_s centered on c because $c > \ln 3$, and $\alpha_1, \alpha_2 > \frac{1}{2} \ln 3$. Similarly, the hexagon containing a contains a closed disk of radius ρ_s centered on a because $\gamma_1, \gamma_2 > \frac{1}{2} \ln 3$.

The remaining case to be considered is when both a and b verify $a, b \leq \ln 3$. In this case, cut along a and b in order to obtain an octagon. On this octagon consider a simple geodesic perpendicular path between one of the copies of a and one of the copies of b, as in Figure 9.

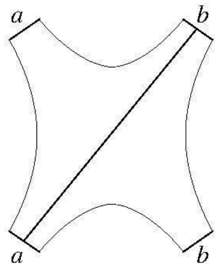

FIGURE 9
When a and b are both short

By cutting along this path one obtains two (in general non-isometric) hexagons. Now by applying Lemma 4.2 to the heights of these hexagons, it is not too difficult to see that all the heights h verify $h > \ln 3$. By once again applying Lemma 4.3, both of these hexagons contain the required disks, and the last case is established.

To show the sharpness of the constant ρ_s on the Y-piece, an example must be constructed that does not allow any disk of radius $\rho > \rho_s$. The example is constructed by pasting two hexagons identical to the one portrayed in Figure 10.

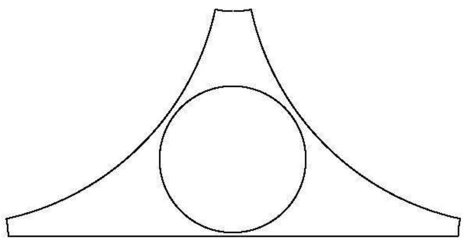

FIGURE 10
A hexagon used to obtain a Y-piece containing the smallest possible disk

To construct a hexagon of this type one must take two l-pentagons and paste them as in Figure 11. The Y-piece is obtained by gluing this hexagon to its mirror image along the edges A, B and C. The boundary edges α, β and γ are of lengths (respectively) $4l$, $2\operatorname{arccosh}(\sinh^2 l)$ and $2\operatorname{arccosh}(\sinh^2 l)$.

For $\rho > \rho_s$ there exists a constant L such that for $l > L$, no disk is contained in the Y-piece. Such objects will be called *l-pants*. The lemma is now proved. □

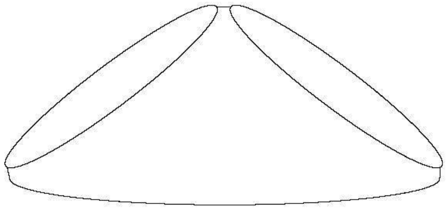

FIGURE 11
An l-pants

The example in the previous proof, as for the l-pentagon, can be generalized. This is the object of the following proposition.

PROPOSITION 4.7. *For $\epsilon > 0$ there exists a value x_ϵ such that for any α, $\beta > x_\epsilon$ there exists another value $x_{\epsilon,\alpha,\beta}$ such that for any $\gamma > x_{\epsilon,\alpha,\beta}$, a Y-piece with boundary geodesics of lengths α, β, γ contains no disk of radius $\rho_s + \epsilon$.*

Note that the value $x_{\epsilon,\alpha,\beta}$ does depend on the choice of α and β.

Proof. The proof is essentially constructive. Consider the symmetric ideal pentagon P (with 4 right angles and the remaining angle 0) as in the following figure.

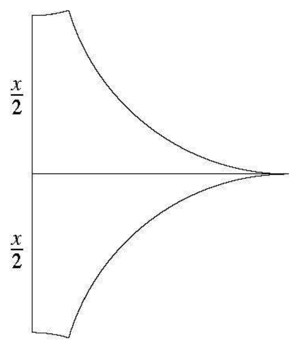

FIGURE 12
A symmetric ideal pentagon

x_ϵ can be so chosen that for all $x > x_\epsilon$ there are no disks of radius $\rho_s + \epsilon$ contained in the interior of P. Now consider the Y-piece obtained with lengths $\alpha, \beta > x_\epsilon$. This Y-piece can be represented as an octagon with lengths as in Figure 13. Notice that, due to the symmetry of Y-pieces, α and β are separated by l_γ into two segments of equal length.

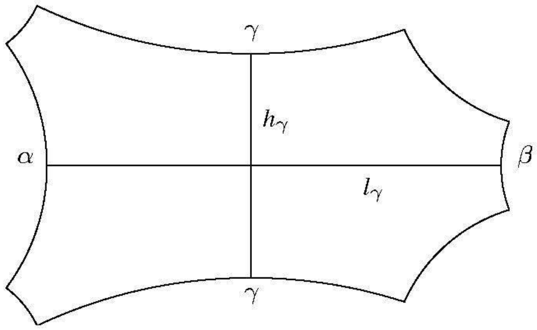

FIGURE 13
A Y-piece cut along two common perpendiculars

Now a value $x_{\epsilon,\alpha,\beta}$ can be chosen large enough so that for all $\gamma > x_{\epsilon,\alpha,\beta}$ the two hexagons separated by h_γ are as close as necessary to the situation represented in Figure 12. This completes the proof. □

There is an immediate corollary to Proposition 4.6.

COROLLARY 4.8. *For S with genus $g \geq 2$, $r_S \geq \frac{1}{2}\ln 3$.* □

Proposition 4.6 is also the central part of the proof of the following

THEOREM 4.9. *Let S be a closed Riemann surface of genus $g \geq 2$. Let $\gamma_1, \ldots, \gamma_{3g-3}$ be disjoint simple closed geodesics on S. There exist at least $4g-4$ disjoint closed disks of radius $\frac{1}{2}\ln 3$ on S such that there is no intersection between the disks and any of the closed geodesics. Conversely, for a given $\rho > \frac{1}{2}\ln 3$ and a genus $g \geq 2$ it is possible to find a closed Riemann surface of genus g with a partition such that a disk of radius ρ always intersects the partition.*

Proof. S can be decomposed into $2g-2$ Y-pieces. By the previous lemma on each Y-piece there exists at least one disk of radius ρ_s and this proves the first part of the theorem. The construction of a closed Riemann surface of genus g which does not admit any open disks of radius ρ for $\rho > \rho_s$ can be done with long l-pants, as in the proof of the previous lemma. By taking $2g-2$ isometric l-pants it is possible to construct a surface of genus g as in Figure 14.

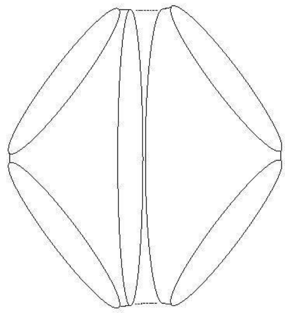

FIGURE 14
A surface obtained by pasting l-pants

The $3g-3$ geodesics are the boundary geodesics of the Y-pieces. □

5. THE EXPLICIT VALUES OF r_S AND r_g

We have shown that our bound for the radius of a disk left untouched by $3g-3$ non intersecting geodesics is sharp, but what can be said if we look at the bound of *one* simple closed geodesic ? Furthermore the sharpness has been proved within the category of closed surfaces of genus g, but the sharpness is not proved for a given individual surface.

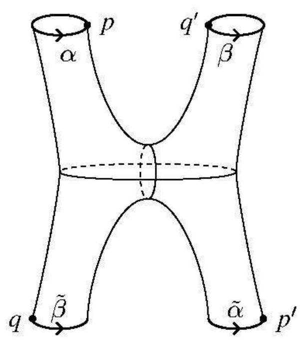

FIGURE 15
Genus 2

The next step is to prove, with an example, that $r_g = \frac{1}{2}\ln 3$. The idea is to construct a surface S and a single geodesic γ that allows only small disks on $S \setminus \gamma$. The example for genus 2 is constructed as follows.

Take eight copies of an l-pentagon (with l large) and paste them together as indicated in Figure 15. What is obtained is an X-piece. Paste together the oriented edges α and $\tilde{\alpha}$ such that p is pasted to p'. Do the same with β and $\tilde{\beta}$ such that q is pasted to q'. Then γ is the geodesic indicated in Figure 16 by a bold curve, following the pasting scheme of Figure 15. For the same reasons as in all previous examples, it is possible for $\rho > \rho_s$ to find a surface of this type with no disks of radius ρ having no intersection with γ.

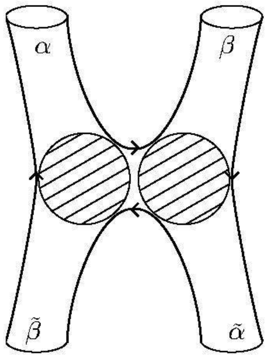

FIGURE 16
The winning geodesic for genus 2

In the same spirit we can construct an example for arbitrary genus g. Take $g-1$ copies of the previously constructed X-piece. Then paste them as in Figure 17.

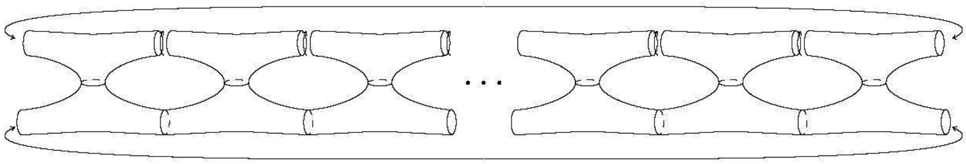

FIGURE 17
Genus g

All arrows indicate a pasting with half twists including the pasting of the four boundary geodesics. The geodesic γ is as indicated in Figure 18 and the surface fulfills our requirements.

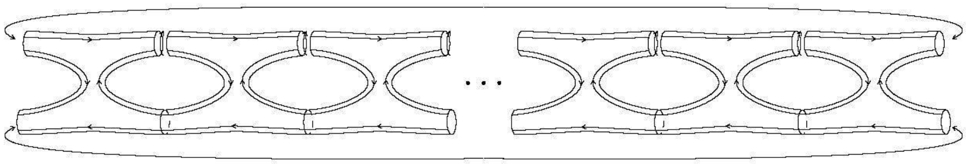

FIGURE 18
The winning geodesic for genus g

This example proves the sharpness of the constant for one closed geodesic within the category of surfaces.

PROPOSITION 5.1. *For* $g \geq 2$, $\rho_g = \frac{1}{2}\ln 3$. □

The final step is to prove, for *any* surface S, that $r_S = \frac{1}{2}\ln 3$. This is done by proving the existence of a simple closed geodesic γ on any surface S so that disks on $S \setminus \gamma$ have maximal radii arbitrarily close to ρ_s. The proof is contained in two lemmas. The first shows that for a given number of disjoint closed simple geodesics, it is possible to construct a simple closed geodesic that imitates all of them, i.e., for an $\epsilon > 0$, it is possible to find a new geodesic γ with $d(\gamma, p) < \epsilon$ for all p on the union of the initial geodesics. We have previously seen that we can construct a genus g surface with a partition with each Y-piece having a maximal radius arbitrarily close to $\frac{1}{2}\ln 3$. The second lemma proves that a partition of this type exists on all surfaces.

LEMMA 5.2. *Let* $\gamma_1, \ldots, \gamma_n$ *be simple closed non-intersecting geodesics on* S. *For* $\epsilon > 0$ *there exists a simple closed geodesic* γ *such that for all* $p \in \gamma_1 \cup \ldots \cup \gamma_n$, $d(p, \gamma) < \epsilon$.

Proof. First complete the set of γ_i's into a partition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ (if necessary). Let γ be a simple closed geodesic such that $\mathrm{int}(\gamma, \gamma_i) \neq 0$ for all $i \in 1, \ldots, 3g-3$. It is a quick topological exercise to see that such a curve always exists, regardless of the nature of the partition. Let $\mathcal{X}_{\gamma_i}$ be the X-piece around γ_i whose boundary curves are elements of $\mathcal{P}$, when such an X-piece exists. (The other case to consider is when γ_i is found inside a one holed torus where the boudary curve is an element of $\mathcal{P}$. The proof in this second case is truly identical, and will be left to the dedicated reader.) $\gamma \cap \mathcal{X}_{\gamma_i}$ is the union of at least 2 disjoint geodesic arcs. At least one of these paths crosses γ_i.

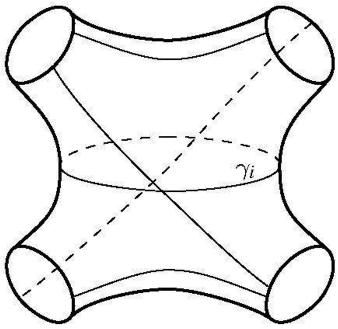

FIGURE 19
$\mathcal{X}_{\gamma_i}$

Let c be one of the two paths. Performing positive Dehn twists around γ_i will eventually increase the length of $\mathcal{D}_{k,\gamma_i}(\gamma)$. In particular the length of the paths c_k, the image of c by k Dehn twists, will increase as well (c_k denotes a path in $\mathcal{D}_{k,\gamma_i}(\gamma) \cap \mathcal{X}_{\gamma_i}$ that crosses γ_i). Let $d_k = \max_{x \in \gamma_i} d_{\mathcal{X}_{\gamma_i}}(x, \mathcal{D}_{k,\gamma_i}(\gamma))$. Let q_k be an intersection point of c_k and γ_i. In the following figure, multiple copies of $\mathcal{X}_{\gamma_i}$, each represented by 4 hexagons, are portrayed, so as to see how c_k evolves when k increases. Notice that the following procedure works for any geodesic path from "start" (the boundary curve through which c enters $\mathcal{X}_{\gamma_i}$) to "finish" (the boundary curve through which c leaves) as shown in the figure. The top hexagons are all isometric, as are the bottom ones. θ_k is the angle between c_k and γ_i. In Figure 20, h_0 is the distance from the "entry point" of c_k on $\mathcal{X}_{\gamma_i}$ and γ_i, and h_1 is the distance from the "exit point" of c_k on $\mathcal{X}_{\gamma_i}$ and γ_i.

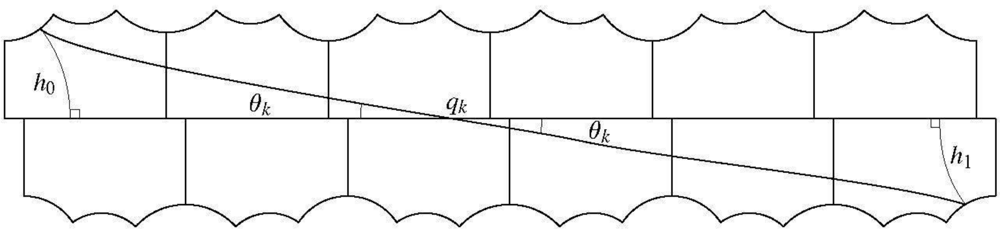

FIGURE 20
$\mathcal{X}_{\gamma_i}$ seen 3 times

Observe the right-angled triangle with vertex q_k and edge h_0. (Observing the right-angled triangle with h_1 instead works as well.) The value for h_0 is always bounded, but the distance between q_k and h_0 grows without bound. This shows that θ_k tends to 0 as k increases. It is then easy to see that

$d_k \geq \max_{x \in \gamma_i} d_S(c_k, x)$ also tends to 0 as k increases. Choose k_i so that for all $k > k_i$, $d_S(c_k, \gamma_i) < \epsilon$ for *any geodesic path* c_k from start to finish.

The process can then be repeated by performing Dehn twists on all the other γ_i's. Choose $k_\epsilon = \max_{i \in \{1,\dots,3g-3\}} k_i$. If we start with γ_1 and finish with γ_{3g-3}, the result obtained is the following simple closed geodesic:

$$\gamma_\epsilon = \mathcal{D}_{k_\epsilon, \gamma_{3g-3}} \circ \mathcal{D}_{k_\epsilon, \gamma_{3g-4}} \dots \circ \mathcal{D}_{k_\epsilon, \gamma_1}(\gamma).$$

Such Dehn twists do not change the nature of the original geodesic (number of intersection points with each γ_i etc.) and the geodesic obtained meets the requirement of the lemma. □

There is an immediate and interesting corollary to this lemma.

COROLLARY 5.3. *Let S be a closed surface of genus g. For any $\epsilon > 0$ there exists a simple closed geodesic δ_ϵ such that for any simple closed geodesic γ we have $d_S(\gamma, \delta_\epsilon) < \epsilon$.*

Proof. It is easy to see that the geodesics of a partition $\mathcal{P}$ intersect all simple closed geodesics of the surface. For $\epsilon > 0$, the lemma proves the existence of a simple closed geodesic δ_ϵ that is of distance inferior to ϵ to all points of the geodesics of the partition. For a given γ let p be an intersection point with the geodesics of $\mathcal{P}$. Thus δ_ϵ is of distance inferior to ϵ from p, and we can conclude that $d_S(\delta_\epsilon, \gamma) < \epsilon$. □

REMARK 5.4. If S is a surface of signature (g, n), then the previous corollary is not true. This is because a boundary geodesic α is not intersected by another simple closed geodesic, and thus all other simple closed geodesics are of distance superior to $\operatorname{arcsinh}(1/\sinh\frac{\alpha}{2})$. However, with the same arguments, the following is true: for given $\epsilon > 0$ and (g, n), there exists S_ϵ of signature (g, n) and δ_ϵ a simple closed geodesic on S_ϵ with the property of the lemma.

The next lemma shows that it is possible to find $3g - 3$ non intersecting geodesics such that S contains disks which do not intersect these geodesics with maximal radius as close as possible to $\frac{1}{2}\ln 3$.

LEMMA 5.5. *For a given S and $\epsilon > 0$ it is possible to find a partition of S such that each Y-piece in the decomposition contains disks of maximal radius smaller than $\frac{1}{2}\ln 3 + \epsilon$.*

Proof. The proof of this lemma imitates that of Proposition 4.7. Let $\mathcal{P}$ be a partition of S and let $\mathcal{Y} \in \mathcal{P}$. In the first section, it was shown that a boundary element of $\mathcal{Y}$ can be replaced by a new arbitrarily long boundary geodesic. Using Proposition 4.7, $\mathcal{Y}$ can be replaced by a Y-piece $\mathcal{Y}'$ which does not contain any disk of radius $\rho_s + \epsilon$. Applying this process to each Y-piece in $\mathcal{P}$ completes the proof. □

The previous two lemmas and Theorem 4.9 yield the final result.

THEOREM 5.6. *Let S be a closed Riemann surface of genus $g > 1$, endowed with a metric of constant curvature -1. Let γ be a simple closed geodesic on S. The set $S \setminus \gamma$ contains $4g-4$ closed disks of radius $\frac{1}{2}\ln 3$. Conversely, if $\rho > \frac{1}{2}\ln 3$ is a given constant, there exists a simple closed geodesic γ_ρ on S such that the set $S \setminus \gamma_\rho$ does not contain any open disk of radius ρ. This implies that $r_S = r_g = \frac{1}{2}\ln 3$.* □

REMARK 5.7. As mentioned in the introduction, this result can be proved using well-known results concerning *laminations*, see for example [5] or [6]. A lamination is a disjoint union of complete simple geodesics (not necessarily closed) and is thus a generalization of a disjoint union of simple closed geodesics. The first part of the theorem can be deduced from the result that a lamination can be completed into a *maximal* lamination, i.e., a lamination whose complement is a set of ideal hyperbolic triangles. The second part could be proved by using iterations of a simple closed geodesic by a pseudo-Anosov map, but once again, the intent of the author is to give an elementary proof of these facts without the use of lamination machinery.

6. MARDEN'S UNIVERSAL CONSTANT

Compact surfaces with constant negative curvature seem to be endowed with a certain number of natural universal constants. In [9], Marden proved a result for Fuchsian groups that is equivalent to the following

THEOREM 6.1. *There exists $r > 0$ with the following property. Let S be a hyperbolic Riemann surface without boundary. There exists a point $x \in S$ such that $D(x, r)$ is simply connected.*

In [10], A. Yamada proved that $r \geq \ln \frac{2+\sqrt{7}}{\sqrt{3}}$ with equality occurring when S is the thrice punctured sphere (which proves that Yamada's lower bound is sharp). Interestingly, $\ln \frac{2+\sqrt{7}}{\sqrt{3}} = \operatorname{arcsinh}(\frac{2}{\sqrt{3}})$ and $\frac{1}{2}\ln 3 = \operatorname{arccosh}(\frac{2}{\sqrt{3}})$. The link is stronger than this apparent coincidence.

First of all, a thrice punctured sphere can be constructed by pasting two ideal hyperbolic triangles along all three edges. The three punctures are the points at infinity. The value $\frac{1}{2}\ln 3$ was also obtained using this triangle as the maximal radius of an inscribed disk. Theorem 5.6, seen in the light of Theorem 6.1, reads as follows:

THEOREM 6.2. *Let S be a hyperbolic Riemann surface of signature (g, n). There exists a point $x \in S$ such that $D(x, \frac{1}{2}\ln 3)$ is simply connected. The value $\frac{1}{2}\ln 3$ is sharp.* □

Notice that in contrast to Theorem 5.6, the value $\frac{1}{2}\ln 3$ is sharp for the set of surfaces with boundary but not for any individual surface. Surfaces with boundary play an important role in a variety of subjects, including the study of Klein surfaces (orientable or non-orientable hyperbolic surfaces). In other words, a Klein surface is either a hyperbolic Riemann surface, or the quotient of a closed hyperbolic Riemann surface by an orientation reversing involution (whose fixed point set is a set of disjoint simple closed geodesics). In terms of Klein surfaces, Theorem 5.6 implies the following corollary, where again the adjective "sharp" means sharp for the set of Klein surfaces.

COROLLARY 6.3. *Let S be a hyperbolic Klein surface. There exists a point $x \in S$ such that $D(x, \frac{1}{2}\ln 3)$ is simply connected. The value $\frac{1}{2}\ln 3$ is sharp.*

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