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SEVERAL ARITHMETIC PROGRESSIONS OF EQUAL LENGTHS AND EQUAL PRODUCTS OF TERMS

by Ajai CHOUDHRY

ABSTRACT. The problem of finding two arithmetic progressions of equal lengths of positive integers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that the products of their terms are equal has been solved for any arbitrary value of n. This paper deals with the hitherto unsolved problem of finding three or more arithmetic progressions of equal lengths and equal products of terms. When $n = 3$ we give a method of generating an arbitrarily large number of arithmetic progressions of positive integers with equal products of terms, and when $n = 4$ we obtain examples of five arithmetic progressions with equal products of terms, three of the five arithmetic progressions consisting only of positive integers.

1. INTRODUCTION

The problem of finding two arithmetic progressions of equal lengths of positive integers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that the products of their terms are equal has been considered by several mathematicians ([2], [3], [4], [5], [7]) and solutions are known for any arbitrary value of n. Till now, however, examples of more than two arithmetic progressions of equal lengths and equal products of terms have not been published.

In this paper we give, for any arbitrary positive integer m howsoever large, a method of generating m arithmetic progressions of three terms each and consisting of positive integers only such that the product of the three terms of all the arithmetic progressions is the same. Further we obtain infinitely many examples of five arithmetic progressions of four terms each such that the products of the four terms in each of the five arithmetic progressions is the same. While three of the five arithmetic progressions may consist of positive integers only, the other two involve negative integers.

We note that if a solution to our problem is found in rational numbers, a solution in integers is readily determined by multiplying all of the terms in all the arithmetic progressions by a suitable integer. In Section 2 we will find arithmetic progressions with three terms each and with equal products of terms while Section 3 deals with arithmetic progressions of four terms each and having a similar property.

2. Several arithmetic progressions of three terms with equal products of terms

LEMMA. *For any arbitrary arithmetic progression* a_1, a_2, a_3, *there exist two associated arithmetic progressions, namely,* $a_3, -a_1/2, -2a_2$, *and* $a_1, -a_3/2, -2a_2$, *which have the same product of terms, that is,* $a_1a_2a_3$. *Further, if the common product* $a_1a_2a_3$ *is positive, one of the three arithmetic progressions consists of positive terms only while the other two consist of one positive term and two negative terms.*

Proof. Since a_1, a_2, a_3 are in arithmetic progression, we have $a_1 - 2a_2 + a_3 = 0$, and it now readily follows that the terms of the two other arithmetic progressions stated in the lemma are actually in arithmetic progression. The product of the terms of all the three arithmetic progressions is easily seen to be $a_1a_2a_3$. Further, if the common product $a_1a_2a_3 > 0$, then either all the three terms of the given arithmetic progression are positive, or two of the terms are negative and one term is positive. If a_1, a_2, a_3 are not all positive and the terms are in ascending order, then $a_1 < 0, a_2 < 0, a_3 > 0$, and so the first associated arithmetic progression consists entirely of positive terms, while if a_1, a_2, a_3 are in descending order, the second associated arithmetic progression is similarly seen to consist of positive terms only. It is also easily seen that in each case two of the three arithmetic progressions consist of one positive term and two negative terms. This completes the proof.

We will now show how to find an arbitrarily large number of arithmetic progressions of three terms each with equal products of terms, the common product being positive. If the terms of some of the arithmetic progressions are not all positive, we will use the above lemma to replace each such arithmetic progression with an associated arithmetic progression consisting of positive terms only. We will thus obtain arbitrarily many arithmetic progressions of three positive terms and with equal products of terms.

We write $f(x, y) = x(x^2 - y^2)$, and note the following identity which may be readily verified by direct computation:

$$f(u_1, v_1) = f(u_2, v_2), \tag{2.1}$$

where

$$u_2 = -\frac{4u_1v_1^2}{9u_1^2 - v_1^2}, \qquad v_2 = \frac{27u_1^4 - 18u_1^2v_1^2 - v_1^4}{2(9u_1^2 - v_1^2)v_1}. \tag{2.2}$$

In the identity (2.1), u_1 and v_1 are arbitrary non-zero parameters, and hence we can replace them by u_2 and v_2 defined above, to get the identity

$$f(u_2, v_2) = f(u_3, v_3), \tag{2.3}$$

where

$$\begin{aligned} u_3 = &-16(27u_1^4 - 18u_1^2v_1^2 - v_1^4)^2u_1v_1^2 \\ &\times \{(729u_1^8 - 972u_1^6v_1^2 + 270u_1^4v_1^4 - 540u_1^2v_1^6 + v_1^8)(9u_1^2 - v_1^2)\}^{-1}, \end{aligned}$$

$$\begin{aligned} v_3 = (&531441u_1^{16} - 1417176u_1^{14}v_1^2 + 1338444u_1^{12}v_1^4 + 367416u_1^{10}v_1^6 \\ &- 1115370u_1^8v_1^8 + 328536u_1^6v_1^{10} - 67284u_1^4v_1^{12} + 1224u_1^2v_1^{14} + v_1^{16}) \\ &\times \{4v_1(9u_1^2 - v_1^2)(27u_1^4 - 18u_1^2v_1^2 - v_1^4)\}^{-1} \\ &\times (729u_1^8 - 972u_1^6v_1^2 + 270u_1^4v_1^4 - 540u_1^2v_1^6 + v_1^8)^{-1}. \end{aligned}$$

In fact we can repeat this process any number of times to get an arbitrarily long diophantine chain of the type

$$f(u_1, v_1) = f(u_2, v_2) = f(u_3, v_3) = \cdots = f(u_m, v_m), \tag{2.4}$$

where m is an arbitrary positive integer and u_i, v_i $(i = 2, 3, \ldots, m)$ are rational homogeneous functions of the arbitrary parameters u_1 and v_1.

We choose u_1 and v_1 to be positive rational numbers such that $u_1 > v_1$. In view of (2.4), all the m arithmetic progressions $u_i - v_i, u_i, u_i + v_i$ $(i = 1, 2, \ldots, m)$ have the same product of terms, namely, $u_1(u_1^2 - v_1^2)$. Since this common product is positive, the terms of each arithmetic progression are either all positive, or, if two of the terms of some of the m arithmetic progressions are negative, these can be replaced by associated arithmetic progressions consisting only of positive terms by applying the lemma.

We now prove that the above procedure actually generates m distinct arithmetic progressions. For any i $(i = 1, 2, 3, \ldots, m)$, the common differences of the i^{th} arithmetic progression and its associated arithmetic progressions may be written as v_i, v_i', v_i'', all of which are homogeneous functions of u_1 and v_1.

When $i \neq j$, the condition that any of the functions v_i, v'_i, v''_i is equal to any of the functions v_j, v'_j, v''_j is expressed by nine polynomial equations in u_1/v_1, and these equations have only a finite number of roots. Further for any given m, there are only finitely many choices for i and j, and thus by choosing u_1 and v_1 suitably so as to avoid the finite number of values of u_1/v_1 that can make the common differences of two of the arithmetic progressions equal, we are assured of generating arbitrarily many distinct arithmetic progressions of three positive terms each with equal products of terms. A similar argument shows that by a suitable choice of u_1 and v_1, we are actually assured of m arithmetic progressions of positive rational numbers, all the terms of all the m arithmetic progressions being distinct, and the m arithmetic progressions having equal product of terms.

The m arithmetic progressions consisting of positive rational numbers and with equal product of terms can now be multiplied by a suitable positive integer to get a solution in positive integers. Further, for any given m, infinitely many examples of arithmetic progressions with the desired property can clearly be generated using the above procedure.

As a numerical example, when $m = 3$ and taking $u_1 = 2$, $v_1 = 1$, we get the following three arithmetic progressions with equal products of terms:

$$(12723520040, 25447040080, 38170560120),$$
$$(5816466304, 34080857250, 62345248196),$$
$$(22560173250, 23124766049, 23689358848).$$

As the above procedure generates arithmetic progressions involving large integers, we give below an identity that generates the desired examples of three arithmetic progressions in much smaller integers. We have

$$f(x_1, y_1) = f(x_2, y_2) = f(x_3, y_3), \tag{2.5}$$

where

$$\begin{aligned}
x_1 &= -144a^2 + 384ab - 192b^2, \\
y_1 &= -3231a^2 + 8616ab - 5673b^2, \\
x_2 &= 1260a^2 - 3672ab + 2668b^2, \\
y_2 &= 630a^2 - 2538ab + 2192b^2, \\
x_3 &= 1260a^2 - 3048ab + 1836b^2, \\
y_3 &= 630a^2 - 822ab - 96b^2,
\end{aligned} \tag{2.6}$$

where a and b are arbitrary parameters. The identity $f(x_1, y_1) = f(x_2, y_2)$ may be obtained by following the method described in [1], and it can also be verified by direct computation. In this identity, if we replace a by $-a+8b/3$, we obtain the identity $f(x_1, y_1) = f(x_3, y_3)$, and combining the two results, we get the identity (2.5).

As a numerical example, taking $a = -1$, $b = 1$ in (2.6), we get three arithmetic progressions $(x_i - y_i, x_i, x_i + y_i)$, $i = 1, 2, 3$, with equal products of terms, and applying the lemma leads to the following three arithmetic progressions in positive integers with equal products of terms:

$$(560, 1900, 3240)\,, \quad (360, 2280, 4200)\,, \quad (1197, 1536, 1875)\,.$$

3. Several arithmetic progressions of four terms with equal products of terms

We will first find two sets of three arithmetic progressions in parametric terms such that in both sets the product of the four terms of each of the arithmetic progressions is equal. By a suitable choice of parameters and appropriate scaling, we will obtain two sets of three arithmetic progressions such that one of the arithmetic progressions is common to both the sets so that we will eventually get five arithmetic progressions such that the product of the four terms in each of the five arithmetic progressions is the same.

If $u-3v, u-v, u+v, u+3v$ is taken as an arithmetic progression with four terms, another arithmetic progression $x-3y, x-y, x+y, x+3y$ will have the products of its terms equal to the product of the terms of the first arithmetic progression if

$$(x^2-y^2)(x^2-9y^2) = (u^2-v^2)(u^2-9v^2)\,. \tag{3.1}$$

To solve (3.1), we write

$$x = t+u\,, \qquad y = -t+v\,, \tag{3.2}$$

when (3.1) reduces to

$$t(u+v)\{4t^2 + (u-11v)t - u^2 - 4uv + 9v^2\} = 0\,. \tag{3.3}$$

The last factor of (3.3) is a quadratic in t, and its discriminant $17u^2+42uv-23v^2$ is easily seen to become a perfect square if we choose

$$\begin{aligned} u &= -59r^2 + 46r - 23\,, \\ v &= 17r^2 - 34r - 19\,, \end{aligned} \tag{3.4}$$

where r is an arbitrary rational parameter, and then we solve (3.3) to get two non-zero rational solutions for t and on using (3.2) we obtain two solutions for

x and y. Thus when u, v are defined by (3.4), we obtain two new arithmetic progressions such that the product of their terms is the same as the product of the terms of the arithmetic progression $u-3v, u-v, u+v, u+3v$, and all the three arithmetic progressions are expressed in terms of the arbitrary parameter r. These three arithmetic progressions with equal products of terms are given below:

(i) arithmetic progression with first term $110r^2-148r-34$ and common difference $-34r^2+68r+38$ — the four terms may be written as

$$2(5r+1)(11r-17), \quad 4(19r-1)(r-1), \quad 42r^2-12r+42, \quad 8(r+5)(r+2);$$

(ii) arithmetic progression with first term $38r^2+188r-10$ and common difference $2r^2-100r+26$ — the four terms may be written as

$$2(r+5)(19r-1), \quad 8(r+2)(5r+1), \quad 42r^2-12r+42, \quad 4(r-1)(11r-17);$$

(iii) arithmetic progression with first term $-64r^2-64r+128$ and common difference $53r^2+26r-43$ — the four terms may be written as

$$-64(r+2)(r-1), \quad -(r+5)(11r-17), \quad 42r^2-12r+42, \quad (5r+1)(19r-1).$$

It is easy to see that for all values of r, some of the terms in the above three arithmetic progressions will be of opposite signs, so we cannot get three arithmetic progressions consisting only of positive integers by giving a suitable value to the parameter r. We however note that when $r<-5$ or $r>17/11$, then all the terms of the first two arithmetic progressions are positive.

To obtain the second set of three arithmetic progressions of four terms each and with equal products of terms, we substitute in (3.1) as follows:

$$x=-3t+u, \qquad y=t+v, \tag{3.5}$$

and proceeding on the same lines as before, we obtain the second set of three arithmetic progressions with equal products of terms. These three arithmetic progressions are given below in terms of an arbitrary rational parameter s:

(i) arithmetic progression with first term $51s^2+666s-1485$ and common difference $69s^2-138s+837$;

(ii) arithmetic progression with first term $-210s^2-1404s-1458$ and common difference $156s^2+552s+828$;

(iii) arithmetic progression with first term $-486s^2+108s+378$ and common difference $248s^2+48s+216$.

Here again for any arbitrary value of s, some of the terms in the three arithmetic progressions will be of opposite signs.

We will now choose the parameters r and s such that the terms of the third arithmetic progression of the first set become proportional to the corresponding terms of the third arithmetic progression of the second set. This will happen when the ratio of the initial terms of these two arithmetic progressions is the same as the ratio of their common differences, for which the condition is

$$\frac{-64r^2 - 64r + 128}{-486s^2 + 108s + 378} = \frac{53r^2 + 26r - 43}{248s^2 + 48s + 216}. \tag{3.6}$$

This condition reduces to the following quadratic equation in r:

$$\begin{aligned}(4943s^2 - 4398s - 16929)r^2 - (1618s^2 + 2940s + 11826)r \\ + 5423s^2 + 5394s + 21951 = 0,\end{aligned} \tag{3.7}$$

and this will have a rational solution for r if the discriminant of the quadratic in r is a perfect square, that is, if there exist rational numbers s and t such that

$$-181607s^4 - 3012s^3 + 130230s^2 + 1425276s + 2823417 = t^2. \tag{3.8}$$

The transformation

$$\begin{aligned} s &= -\frac{3(161X + 2Y - 26096)}{(29X - 6Y - 76976)}, \\ t &= \frac{18432(4X^3 - 3639X^2 - 1540410X - 90909Y - 342830128)}{(29X - 6Y - 76976)^2}, \end{aligned} \tag{3.9}$$

reduces (3.8) to the elliptic curve

$$Y^2 = X^3 + X^2 + 384496X - 15445440. \tag{3.10}$$

A rational point on the curve (3.10) is (1213/4, 90909/8) and since this point does not have integer co-ordinates, it follows from the Nagell-Lutz theorem on elliptic curves [6, p. 56] that this is not a point of finite order. Thus there exist infinitely many rational points on the elliptic curve (3.10) and these can be obtained by the group law. These infinitely many rational points on the curve (3.10) yield infinitely many rational solutions of equation (3.8). Each of the infinitely many values of s so obtained ensures that equation (3.7) has rational solutions for r, and with these infinitely many values of r and s, the terms of the third arithmetic progression of the first set become proportional to the corresponding terms of the third arithmetic progression of the second set. We can now multiply the arithmetic progressions of both sets by suitable integers to get two sets of three arithmetic progressions such that the third arithmetic progression in both sets is identical. The product of the terms of the arithmetic progressions in each set is now the same and we thus obtain five arithmetic progressions with this property.

It is easily seen that two out of the five arithmetic progressions obtained above can be identical only for a finite set of values of r and s. Since we can obtain infinitely many values of r and s that lead to five arithmetic progressions with the desired property, we can simply exclude the finite set of values that make two arithmetic progressions identical, and we are thus assured of generating infinitely many examples of five distinct arithmetic progressions such that the product of their terms is the same.

As a numerical example, the rational point (3784, 235872) on the curve (3.10) yields the solution $(r, s) = (79/37, 483/211)$ of equation (3.7) which leads to the following five arithmetic progressions with equal products of terms:

$$(360, 427, 494, 561), \quad (70, 494, 918, 1342), \quad (114, 442, 770, 1098),$$
$$(-714, -110, 494, 1098), \quad (-2145, -1064, 17, 1098).$$

Similarly the rational point (70840/81, 22943440/729) on the curve (3.10) yields the solution $(r, s) = (30151/15421, 42941/19373)$ of (3.7) which leads to the following five arithmetic progressions with equal products of terms:

$$(40103808, 57022285, 73940762, 90859239),$$
$$(7109680, 73940762, 140771844, 207602926),$$
$$(13633284, 62696772, 111760260, 160823748),$$
$$(-315867444, -156970380, 1926684, 160823748),$$
$$(-99825210, -12942224, 73940762, 160823748).$$

In both of the above examples, there are three arithmetic progressions consisting entirely of positive integers. More such examples in which three of the five arithmetic progressions consist only of positive integers can be found and it seems that there exist infinitely many examples of three arithmetic progressions of four terms consisting of positive integers such that the products of the terms are equal.

It would be interesting to find three arithmetic progressions of positive integers having five or more terms such that the products of their terms are equal but so far no such examples have been found.

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