The combinatorial cost

L'Enseignement Mathématique (2) 53 (2007), 225–235

THE COMBINATORIAL COST

by Gábor ELEK *)

ABSTRACT. We study the combinatorial analogues of the classical invariants of measurable equivalence relations. We introduce the notion of cost and β-invariants (the analogue of the first L^2-Betti number introduced by Gaboriau [3]) for sequences of finite graphs with uniformly bounded vertex degrees and examine the relation of these invariants and the rank gradient resp. mod p homology gradient invariants introduced by Lackenby ([5], [6]) for residually finite groups.

1. INTRODUCTION

1.1 GRAPH SEQUENCES

Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a sequence of finite simple graphs satisfying the following conditions:

- $\sup_{1\leq n<\infty} \max_{x\in V(G_n)} \deg(x) < \infty$. That is, the graphs have uniformly bounded vertex degrees.
- $|V(G_n)| \to \infty$ as $n \to \infty$.

In the sequel we refer to such systems as *graph sequences*. Now let $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ be another graph sequence such that $V(H_n) = V(G_n)$ for any $n \geq 1$. Then $\mathcal{H} \prec \mathcal{G}$ if there exists an integer $L > 0$ such that for any $n \geq 1$ and $x, y \in V(H_n)$, $d_{G_n}(x,y) \leq L\, d_{H_n}(x,y)$, where d_{G_n} resp. d_{H_n} denote the shortest path metrics on G_n resp. on H_n. That is, if x and y are adjacent in the graph H_n then there exists a path between x and y in G_n of length at most L. We say that $\mathcal{G}$ and $\mathcal{H}$ are *equivalent*, $\mathcal{G} \simeq \mathcal{H}$, if $\mathcal{H} \prec \mathcal{G}$ and $\mathcal{G} \prec \mathcal{H}$. The *edge measure* of $\mathcal{G}$ is defined as

$$e(\mathcal{G}) := \liminf_{n\to\infty} \frac{|E(G_n)|}{|V(G_n)|}$$

*) The author is supported by OTKA Grants T 049841 and T 037846.

and the *cost* of $\mathcal{G}$ is given as

$$c(\mathcal{G}) := \inf_{\mathcal{H} \simeq \mathcal{G}} e(\mathcal{H}).$$

Clearly, $c(\mathcal{G}) \geq 1$ for any graph sequence $\mathcal{G}$. Originally, the cost was defined for measurable equivalence relations by Levitt [7]. In our paper we view graph sequences as the analogues of L-graphings of measurable equivalence relations (see [4]).

Recall that a graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is a *large girth sequence* if for any $k \geq 1$, there exists n_k such that if $n \geq n_k$ then G_n does not contain a cycle of length not greater than k. Large girth sequences are the analogues of L-treeings [4]. Our first goal is to prove the following version of Gaboriau's Theorem [2], (see also [4], Theorem 19.2).

THEOREM 1.1. *If $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is a large girth sequence, then* $e(\mathcal{G}) = c(\mathcal{G})$.

1.2 β-INVARIANTS

In the proof of Theorem 1.1 we shall use the β-invariants which are the analogues of the first L^2-Betti numbers of measurable equivalence relations [3]. First recall the notion of cycle spaces.

Let $G(V,E)$ be a finite, simple, connected graph and K be a commutative field. Let $\varepsilon_K(G)$ be the vector space over K spanned by the edges and let $C_K(G) \subseteq \varepsilon_K(G)$, the *cycle space*, be the subspace generated by the cycles of G. Then $\dim_K C_K(G) = |E| - |V| + 1$. Now let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a graph sequence. Let $C_K^q(G_n)$ be the space spanned by the cycles of G_n of length not greater than q. Here we use the usual convention that $(x,y) = -(y,x)$ and we associate to the cycle $(x_1, x_2, \dots, x_n, x_1)$ the vector $\left(\sum_{i=1}^{n-1}(x_i, x_{i+1}) + (x_n, x_1)\right)$.

Set

$$s_K^q(\mathcal{G}) := \liminf_{n\to\infty} \frac{|E(G_n)| - \dim_K C_K^q(G_n)}{|V(G_n)|} - 1.$$

The β_K*-invariant* of $\mathcal{G}$ is defined as

$$\beta_K(\mathcal{G}) := \inf_q s_K^q(\mathcal{G}).$$

In Section 2 we shall prove that if $\mathcal{G} \simeq \mathcal{H}$, then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$. This immediately shows that

$$\beta_K(\mathcal{G}) + 1 \leq c(\mathcal{G}).$$

1.3 Residually finite groups

Let Γ be a finitely generated group and

$$\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \dots, \qquad \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$$

be a nested sequence of finite index normal subgroups. Following Lackenby [5] we define the *rank gradient* of the system $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$

$$\text{rk grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = \lim_{n\to\infty} \frac{d(\Gamma_n)}{|\Gamma : \Gamma_n|},$$

where $d(\Gamma_n)$ is the minimal number of generators for Γ_n. In another paper [6], Lackenby investigated the behaviour of the sequence $\left\{\frac{d_p(\Gamma_n)}{|\Gamma:\Gamma_n|}\right\}_{n=1}^{\infty}$, where $d_p(\Gamma_n) = \dim_{\mathbf{F}_p} H_1(\Gamma_n, \mathbf{F}_p)$. Here we denote by $\mathbf{F}_p$ the finite field of p elements. Note that $d_p(\Gamma_n) \leq d(\Gamma_n)$. The *mod-$p$-homology gradient* of the system $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$ is defined as

$$p\text{-grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = \liminf_{n\to\infty} \frac{d_p(\Gamma_n)}{|\Gamma : \Gamma_n|}.$$

Let S be a symmetric generating system for Γ and let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be the graph sequence of the Cayley-graphs of Γ/Γ_n with respect to S. We have the following theorem:

Theorem 1.2. $c(\mathcal{G}) - 1 \leq \text{rk grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$.
If Γ is even finitely presented, then we have the inequality

$$\beta_{\mathbf{Q}}(\mathcal{G}) = \beta^1_{(2)}(\Gamma) \leq p\text{-grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = \beta_{\mathbf{F}_p}(\mathcal{G}) \leq c(\mathcal{G}) - 1,$$

where $\beta^1_{(2)}(\Gamma)$ is the first L^2-Betti number of Γ (see [8]).

1.4 Hyperfinite graph sequences

One of the key notions in the theory of measurable equivalence relations is *hyperfiniteness*. We introduce a similar notion for graph sequences. We shall prove the following analogues of Proposition 22.1 and Lemma 23.2 of [4].

Proposition 1.3.
1. *If $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ is a hyperfinite graph sequence then $c(\mathcal{H}) = 1$.*
2. *For any graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ there exists a hyperfinite graph sequence $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ such that $\mathcal{H} \prec \mathcal{G}$.*

Finally we prove the analogue of the theorem of Connes, Feldman and Weiss ([4], Theorem 10.1).

THEOREM 1.4. *Let Γ be a finitely generated residually finite group with a nested sequence of finite index normal subgroups Γ_n, $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. Then the associated graph sequence $\mathcal{G}$ is hyperfinite if and only if Γ is amenable.*

2. β-INVARIANTS

PROPOSITION 2.1. *Let $\mathcal{G} \simeq \mathcal{H}$ be equivalent graph sequences and K be a field. Then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$.*

Proof. Suppose that $\mathcal{H} \subseteq \mathcal{G}$, that is for any $n \geq 1$, $E(H_n) \subseteq E(G_n)$. Let $L > 0$ be an integer such that $d_{G_n}(x,y) \leq L d_{H_n}(x,y)$. We define a K-linear transformation between quotient spaces:

$$\widetilde{\phi}\colon \varepsilon_K(H_n)/C_K^q(H_n) \to \varepsilon_K(G_n)/C_K^q(G_n)$$

by extending the inclusion $\phi\colon E(H_n) \to E(G_n)$.

LEMMA 2.2. *If $\widetilde{\phi}$ is surjective then $q > L$.*

Proof. Let $e = (x,y) \in E(G_n)$, then there exists a path P between x and y, in H_n of length not greater than L. The cycle $c = P \cup e$ represents an element in $C_K^q(G_n)$ and

$$[e] \in [c] \oplus [\widetilde{\phi}(\varepsilon_K(H_n))] .$$

Hence the lemma follows. $\square$

By the lemma it follows that $s_K^q(H_n) \geq s_K^q(G_n)$ if $q > L$, thus $\beta_K(\mathcal{H}) \geq \beta_K(\mathcal{G})$.

Now we define another K-linear transformation:

$$\widetilde{\psi}\colon \varepsilon_K(G_n)/C_K^q(G_n) \to \varepsilon_K(H_n)/C_K^{qL}(H_n),$$

by mapping the basis vector $e = (x,y) \in E(G_n)$ to a path in H_n of length not greater than L connecting x and y. If $e \in H_n$, then let $\widetilde{\psi}(e) = e$. Obviously, $\widetilde{\psi}$ is surjective therefore $s_K^q(G_n) \geq s_K^{qL}(H_n)$ and consequently $\beta_K(\mathcal{G}) \geq \beta_K(\mathcal{H})$.

Hence if $\mathcal{G} \simeq \mathcal{H}$, $\mathcal{H} \subseteq \mathcal{G}$ then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$. Now we consider the general case, where $\mathcal{H}$ is an arbitrary graph sequence such that $\mathcal{H} \simeq \mathcal{G}$. Then let $\mathcal{J} = \mathcal{G} \cup \mathcal{H}$, that is $V(J_n) = V(G_n), E(J_n) = E(G_n) \cup E(H_n)$. Clearly, $\mathcal{J} \simeq \mathcal{G} \simeq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{J}$, $\mathcal{G} \subseteq \mathcal{J}$. Thus by our argument above, $\beta_K(\mathcal{H}) = \beta_K(\mathcal{J}) = \beta_K(\mathcal{G})$. $\square$

PROPOSITION 2.3. *Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a graph sequence. Then*

$$\beta_{\mathbf{Q}}(\mathcal{G}) \leq \beta_{\mathbf{F}_p}(\mathcal{G}) \leq c(\mathcal{G}) - 1 .$$

Proof. Let $\mathcal{H} \simeq \mathcal{G}$, then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H}) \leq e(\mathcal{H}) - 1$. Therefore $\beta_K(\mathcal{G}) \leq c(\mathcal{G}) - 1$.

LEMMA 2.4. $\dim_{\mathbf{Q}} C^q_{\mathbf{Q}}(G_n) \leq \dim_{\mathbf{F}_p} C^q_{\mathbf{F}_p}(G_n)$.

Proof. Let c^q_n be the number of cycles in G_n of length not greater than q. Let $\rho_{\mathbf{Z}} \colon \mathbf{Z}^{c^q_n} \to \mathbf{Z}^{|E(G_n)|}$ be the homomorphism that maps $\bigoplus_{i=1}^{c^q_n} s_i$ to $\sum_{i=1}^{c^q_n} s_i[c_i]$, where $s_i \in \mathbf{Z}$ and $[c_i]$ is the integer vector generated by the i-th cycle c_i. Similarly, we define $\rho_{\mathbf{F}_p} \colon \mathbf{F}_p^{c^q_n} \to \mathbf{F}_p^{|E(G_n)|}$. Let $\pi_1 \colon \mathbf{Z}^{c^q_n} \to \mathbf{F}_p^{c^q_n}$, $\pi_2 \colon \mathbf{Z}^{|E(G_n)|} \to \mathbf{F}_p^{|E(G_n)|}$ be the residue class maps. Then $\pi_2 \circ \rho_{\mathbf{Z}} = \rho_{\mathbf{F}_p} \circ \pi_1$. Therefore,

$$\operatorname{rank}_{\mathbf{Z}} \operatorname{Im} \rho_{\mathbf{Z}} \geq \dim_{\mathbf{F}_p} \operatorname{Im} \rho_{\mathbf{F}_p} .$$

Clearly, $\operatorname{rank}_{\mathbf{Z}} \operatorname{Im} \rho_{\mathbf{Z}} = \dim_{\mathbf{Q}} C^q_{\mathbf{Q}}(G_n)$ and $\dim_{\mathbf{F}_p} \operatorname{Im} \rho_{\mathbf{F}_p} = \dim_{\mathbf{F}_p} C^q_{\mathbf{F}_p}(G_n)$. Thus our lemma follows. $\square$

By our lemma, $\beta_{\mathbf{Q}}(\mathcal{G}) \leq \beta_{\mathbf{F}_p}(\mathcal{G})$ hence we finish the proof of our proposition. $\square$

QUESTION 2.5. *Does there exist a graph sequence $\mathcal{G}$ for which $\beta_{\mathbf{Q}}(\mathcal{G}) \neq \beta_{\mathbf{F}_p}(\mathcal{G})$ or $\beta_{\mathbf{F}_p}(\mathcal{G}) \neq c(\mathcal{G}) - 1$?*

Finally we prove Theorem 1.1.

Proof. Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a large girth graph sequence. Then by definition $\beta_K(\mathcal{G}) = e(\mathcal{G}) - 1$. That is, $e(\mathcal{G}) - 1 \leq c(\mathcal{G}) - 1$, hence our theorem follows. $\square$

3. RESIDUALLY FINITE GROUPS

The goal of this section is to prove Theorem 1.2. Let Γ be a finitely generated residually finite group with a not necessarily symmetric generating system S. Let $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \dots$, $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$ be a nested sequence of finite index normal subgroups and $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be the graph sequence, where G_n is the (left) Cayley-graph of the finite group Γ/Γ_n with respect to S. Note that if S' is another generating system and $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ is the associated graph sequence then $\mathcal{H} \simeq \mathcal{G}$.

PROPOSITION 3.1. $c(\mathcal{G}) - 1 \leq \operatorname{rk} \operatorname{grad}\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$.

Proof. First note that by the Reidemeister-Schreier theorem the groups Γ_n are finitely generated as well [9], moreover if T is a finite generating system of Γ_n, then

$$d_{G_T^{\Gamma_n}}(x, y) \leq L\, d_{G_S^{\Gamma}}(x, y)$$

for any $x, y \in \Gamma_n$, where G_S^{Γ} resp. $G_T^{\Gamma_n}$ are the Cayley-graphs with respect to S resp. to T, and the Lipschitz constant L depends only on S and T.

LEMMA 3.2. *For any* $k \geq 1$,

$$\frac{d(\Gamma_k)}{|\Gamma : \Gamma_k|} + 1 \geq c(\Gamma).$$

Proof. We use an idea resembling an argument in the proof of Theorem 21.1 of [4]. Let T be a generating system of Γ_k of minimal number of generators. For simplicity we suppose that $T \subset S$. Consider the following graph sequence: $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$, $V(H_n) = \Gamma/\Gamma_n$. If $n \leq k$, let $H_n = G_n$. Set $S_n = \Gamma_k/\Gamma_n$ and let H'_n be the Cayley-graph of S_n with respect to T. Now enumerate the vertices of $V(H_n)\backslash S_n$, $\{x_1, x_2 \dots, x_{r_n}\}$. For each x_i consider the set of shortest paths in G_n from x_i to the set S_n. Pick the minimal path with respect to the lexicographic ordering. The edges of H_n shall consist of H'_n and the edges of the minimal paths. Define a map $\pi \colon V(H_n) \to S_n$ in the following way. For each $x_i \in V(H_n)\backslash S_n$ let $\pi(x_i) \in S_n$ be the endpoint of the minimal path from x_i to S_n and let $\pi(x) = x$ if $x \in S_n$. By the lexicographic minimality, the union of the paths form a subforest in G_n having exactly $|V(H_n)\backslash S_n|$ edges.

We claim that $\mathcal{H} \simeq \mathcal{G}$. Since $\mathcal{H} \subset \mathcal{G}$, we only need to prove that $\mathcal{G} \prec \mathcal{H}$. Let $n > k$, $x, y \in V(G_n)$. Consider the shortest G_n-path from x to y, $\{x_0, x_1, \dots x_l\}$, $x_0 = x$, $x_l = y$. Let us consider the sequence of vertices $\{\pi(x_0), \pi(x_1), \dots \pi(x_l)\}$.

Let $y_1, y_2, \dots, y_{|\Gamma:\Gamma_k|}$ be a set of coset-representatives with respect to Γ_k. Let t be the maximal word-length of the representatives with respect to S. Then $d_{G_n}(\pi(x), x) \leq t$ for any $x \in V(G_n)$. Therefore, $d_{G_n}(\pi(x_i), \pi(x_{i+1})) \leq 2t + 1$. That is, $d_{H_n}(\pi(x_i), \pi(x_{i+1})) \leq L(2t + 1)$, where L is the Lipschitz-constant defined before the statement of our lemma. Consequently,

$$d_{H_n}(x, y) \leq L\,(2t + 1)\, d_{G_n}(x, y)$$

and therefore $\mathcal{H} \simeq \mathcal{G}$.

For the edge measure of $\mathcal{H}$ we have

$$e(\mathcal{H}) = \liminf_{n\to\infty} \frac{|\Gamma : \Gamma_n| - |\Gamma_k : \Gamma_n| + |E(H'_n)|}{|\Gamma : \Gamma_n|}.$$

The vertex degrees of H'_n are not greater than $2|T| = 2d(\Gamma_k)$, also $|S_n| = |\Gamma_k : \Gamma_k|$. Thus

$$c(\mathcal{G}) \leq e(\mathcal{H}) \leq \frac{d(\Gamma_k)}{|\Gamma : \Gamma_k|} + 1.$$

Hence the lemma follows. $\square$

Proposition 3.1 is a straightforward consequence of Lemma 3.2. $\square$

Let $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}, S, \mathcal{G}$ be as above. Moreover suppose that Γ is finitely presented. This means that if $\Theta\colon \mathcal{F}_S \to \Gamma$ is the natural map from the free group generated by S to Γ then $\ker\Theta$ is generated by the relations $\{R_1, R_2, \ldots, R_l\}$ as a normal subgroup, that is, if $\Theta(\underline{w}) = 1$ then

$$\underline{w} = \prod_{j=1}^{r_w} \gamma_j R_{i_j} \gamma_j^{-1}, \quad \gamma_j \in \mathcal{F}_S.$$

Let $\widetilde{\Sigma}$ be the usual covering CW-complex constructed from $\{R_i\}_{i=1}^{l}$, the 1-skeleton of $\widetilde{\Sigma}$ is the Cayley-graph of Γ and for each $\gamma \in \Gamma$ and $1 \leq i \leq l$, we add a 2-cell $\sigma_{\gamma,i}$ such that

$$\partial\sigma_{\gamma,i} = \sum_{j=1}^{s_i} (\underline{w}_j\gamma, \underline{w}_{j-1}\gamma),$$

where $R_i = a_{s_i}a_{s_i-1}\ldots a_2a_1$, $\underline{w}_j = a_ja_{j-1}\ldots a_2a_1$, $\underline{w}_0 = 1$. Then $\widetilde{\Sigma}$ is simply connected with a natural Γ-action. Clearly, $\pi_1(\widetilde{\Sigma}/\Gamma_n) = \Gamma_n$. Recall that the group homology space $H_1(\Gamma_n, K)$ is isomorphic to the CW-homology space $H_1(\widetilde{\Sigma}/\Gamma_n, K)$.

LEMMA 3.3. *We have*

$$\lim_{n\to\infty} \frac{\dim_K H_1(\widetilde{\Sigma}/\Gamma_n, K)}{|\Gamma : \Gamma_n|} = \beta_K(\mathcal{G}).$$

Proof. Consider the homology complex

$$C_2(\widetilde{\Sigma}/\Gamma_n, K) \xrightarrow{\partial_2} C_1(\widetilde{\Sigma}/\Gamma_n, K) \xrightarrow{\partial_1} C_0(\widetilde{\Sigma}/\Gamma_n, K).$$

Observe that

$$C_1(\widetilde{\Sigma}/\Gamma_n, K) \simeq \varepsilon_K(G_n) \quad \text{and} \quad \dim_K C_0(\widetilde{\Sigma}/\Gamma_n, K) = |V(G_n)|\,.$$

Let r be the maximal word-length of a relation R_i. Then $\partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ is generated by cycles of length at most r. On the other hand, for any $q > r$, the q-cycles are in $\partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ if n is large enough.

Therefore $C_K^q(G_n) = \partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ if n is large enough. Consequently,

$$s_K^q(\mathcal{G}) = \liminf_{n\to\infty} \frac{|E(G_n)| - \dim_K \partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K)) - |V(G_n)|}{|\Gamma : \Gamma_n|}\,.$$

On the other hand,

$$\begin{aligned} \frac{\dim_K H_1(\widetilde{\Sigma}/\Gamma_n, K)}{|\Gamma : \Gamma_n|} &= \frac{\dim_K \ker \partial_1 - \dim_K \operatorname{Im} \partial_2}{|\Gamma : \Gamma_n|} \\ &= \frac{|E(G_n)| - \dim_K \partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K)) - |V(G_n)| + 1}{|\Gamma : \Gamma_n|}\,. \end{aligned}$$

Hence the lemma follows. □

Now we prove the second part of Theorem 1.2.

PROPOSITION 3.4. *Let Γ be a finitely presented residually finite group, $\{\Gamma, \{\Gamma_n\}_{n=1}^\infty\}, S, \mathcal{G}$ be as above. Then*

$$\beta_{\mathbf{Q}}(\mathcal{G}) = \beta^1_{(2)}(\Gamma) \leq p\text{-grad}\{\Gamma, \{\Gamma_n\}_{n=1}^\infty\} = \beta_{\mathbf{F}_p}(\mathcal{G}) \leq c(\mathcal{G}) - 1\,,$$

where $\beta^1_{(2)}(\Gamma)$ is the first L^2-Betti number of Γ (see [8]*).*

Proof. By Lemma 3.3, $\beta_{\mathbf{F}_p}(\mathcal{G}) = p\text{-grad}\{\Gamma, \{\Gamma_n\}_{n=1}^\infty\}$. Also,

$$\beta_{\mathbf{Q}}(\mathcal{G}) = \liminf_{n\to\infty} \frac{\dim_{\mathbf{Q}} H_1(\widetilde{\Sigma}/\Gamma_n, \mathbf{Q})}{|\Gamma : \Gamma_n|}\,.$$

By the Approximation Theorem of Lück

$$\lim_{n\to\infty} \frac{\dim_{\mathbf{Q}} H_1(\widetilde{\Sigma}/\Gamma_n, \mathbf{Q})}{|\Gamma : \Gamma_n|} = \beta^1_{(2)}(\Gamma)\,.$$

Hence our proposition follows. □

QUESTION 3.5. 1. *Does there exist a finitely presented residually finite group Γ and a system $\{\Gamma, \{\Gamma_n\}_{n=1}^\infty\}$ such that*

$$\beta^1_{(2)}(\Gamma) \neq p\text{-grad}\{\Gamma, \{\Gamma_n\}_{n=1}^\infty\} \quad \textit{or} \quad p\text{-grad}\{\Gamma, \{\Gamma_n\}_{n=1}^\infty\} \neq c(\mathcal{G}) - 1\,?$$

2. *Does there exist a finitely generated residually finite group* Γ *and a system* $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$ *such that*

$$c(\mathcal{G}) - 1 \neq \operatorname{rk}\operatorname{grad}\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}\,?$$

4. Hyperfinite graph sequences

We say that a graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is *hyperfinite* if for any $\epsilon > 0$ there exists $K_\epsilon > 0$, positive integers $\{k_n\}_{n=1}^{\infty}$ and a sequence of partitions of the vertex sets $V(G_n)$

$$A_1^n \cup A_2^n \cup \cdots \cup A_{k_n}^n = V(G_n)$$

such that

- For any $n \geq 1$, $1 \leq i \leq k_n$, $|A_i^n| \leq K_\epsilon$.
- If E_n^ϵ is the set of edges $(x, y) \in E(G_n)$ such that $x \in A_i$, $y \in A_j$, $x \neq y$, then
$$\liminf_{n\to\infty} \frac{|E_n^\epsilon|}{|V(G_n)|} \leq \epsilon\,.$$

Now we prove Proposition 1.3.

Proof. Suppose that $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is hyperfinite. Let $\mathcal{H}^\epsilon = \{H_n^\epsilon\}_{n=1}^{\infty}$ be the following graph sequence. The vertex set of H_n^ϵ is $V(G_n)$, $E(H_n^\epsilon)$ is the union of E_n^ϵ and a spanning tree for each connected component of the graphs spanned by the vertices of A_i^n, $1 \leq i \leq k_n$. Clearly, $\mathcal{H}^\epsilon \simeq \mathcal{G}$ and $|E(H_n^\epsilon)| \leq |E_n^\epsilon| + |V(G_n)|$ thus $e(\mathcal{H}^\epsilon) \leq 1 + \epsilon$. Therefore $c(\mathcal{G}) = 1$.

Now we show that for any graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$, $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ is hyperfinite where H_n is a spanning tree of G_n. We actually show that a sequence of trees $\mathcal{T} = \{T_n\}_{n=1}^{\infty}$ is always hyperfinite. Let q be an integer and consider a maximal q-net $L_n^q \subset V(T_n)$. That is, if $x \neq y \in L_n^q$ then $d_{T_n}(x, y) \geq q$ and for any $z \in V(T_n)$ there exists $x \in L_n^q$ such that $d_{T_n}(x, z) \leq q$. Now for each $x \in V(T_n)$ let $\pi(x)$ be one of the vertices $y \in L_n^q$ closest to x. Then $\bigcup_{y\in L_n^q} \pi^{-1}(y)$ is a partition of $V(T_n)$. Clearly $|\pi^{-1}(y)| \geq q$ for any $y \in L_n^q$. Obviously the T_n^y subgraph spanned by the vertices in $\pi^{-1}(y)$ is connected. Thus

$$|E_n^\epsilon| \leq |V(T_n)| - (|V(T_n)| - |L_n^q|)\,.$$

Here we used the fact that a connected graph has at least as many edges as the number of its vertices minus one. Obviously, $|L_n^q| \leq \frac{|V(T_n)|}{q}$, therefore

$$\lim_{n\to\infty} \frac{|E_n^c|}{|V(T_n)|} \leq \frac{1}{q}\,.$$

Consequently, the graph sequence $\mathcal{T}$ is indeed hyperfinite. $\square$

Finally, we prove Theorem 1.4.

Proof. First let Γ be a residually finite non-amenable group with a symmetric generating system S and a nested sequence of finite index normal subgroups $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \dots,\ \bigcap_{n=1}^\infty \Gamma_n = \{1\}$. Let G_n be the Cayley-graph of Γ/Γ_n with respect to S and G_S^Γ be the Cayley-graph of the group Γ. Since Γ is non-amenable, it has no Følner-exhaustion, consequently there exists a real number $\delta > 0$ such that for each finite subset $F \subset \Gamma$ the number of edges from F to the complement of F is at least $\delta|F|$. Fix an integer $m > 0$. If n is large enough then for any subset $M \subset \Gamma/\Gamma_n$, $|M| \leq m$ the number of edges from M to its complement must be at least $\delta|M|$. This follows easily form the fact that for any $r \geq 0$, the r-balls in G_n and in G_S^Γ are isometric. This implies that $\mathcal{G}$ is not hyperfinite.

Now let $\Gamma, \{\Gamma_n\}_{n=1}^\infty, S, \mathcal{G}$ be as above, but let Γ be amenable. The following lemma is a straightforward consequence of Theorem 2 of [1].

LEMMA 4.1. *For any $\omega > 0$, there exist $L_\omega > 0$, $M_\omega > 0$ and a sequence of family of subsets*

$$\{W_n^i\}_{i=1}^{k_n}, \quad W_n^i \subset V(G_n) \quad \textit{if} \quad n \geq M_\omega$$

such that for any $1 \leq i \leq k_n$,

- $|W_n^i| \leq L_\omega$,
- $|W_n^i \backslash \bigcup_{j\neq i}^{k_n} W_n^j| \geq (1-\omega)|W_n^i|$,
- *the number of edges from W_n^i to its complement is at most $\omega|W_n^i|$,*

and

- $|\bigcup_{i=1}^{k_n} W_n^i| \geq (1-\omega)|V(G_n)|$.

Now let $Z_n^i = W_n^i \backslash \bigcup_{j\neq i}^{k_n} W_n^j$ and consider the partition of $V(G_n)$,

$$V(G_n) = \bigcup_{i=1}^{k_n} Z_n^i \cup \bigcup_{j=1}^{l_n} T_n^i\,,$$

where T_n^i are arbitrary subsets of size at most L_ω. Let E_n^ω be the set of edges $(x,y) \in G_n$ such that their endpoints belong to different subsets in the partition. There are three kinds of edges in E_n^ω :

- Edges with an endpoint in T_n^i. The number of such edges is at most $2|S|(1-(1-\omega)^2)|V(G_n)|$.
- Edges from Z_n^i to the complement of W_n^i, for some $1 \leq i \leq k_n$. The number of such edges is at most $2|S|\omega(1-\omega)^{-1}|V(G_n)|$.
- Edges from Z_n^i to $W_n^i \backslash Z_n^i$ for some $1 \leq i \leq k_n$. The number of such edges is at most $2|S|\omega(1-\omega)^{-1}|V(G_n)|$.

Hence

$$\liminf_{n\to\infty} \frac{|E_n^\omega|}{|V(G_n)|} \leq 2|S|\big((1-(1-\omega)^2) + 2\omega(1-\omega)^{-1}\big).$$

Therefore $\mathcal{G}$ is hyperfinite. □

REFERENCES

[1] ELEK, G. The strong approximation conjecture holds for amenable groups. *J. Funct. Anal. 239* (2006), 345–355.

[2] GABORIAU, D. Coût des relations d'équivalence et des groupes. *Invent. Math. 139* (2000), 41–98.

[3] —— Invariants l^2 de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci. 95* (2002), 93–150.

[4] KECHRIS, A. S. and B. D. MILLER. *Topics in Orbit Equivalence Theory.* Lecture Notes in Mathematics *1852*. Springer-Verlag, Berlin, 2004.

[5] LACKENBY, M. Expanders, rank and graphs of groups. *Israel J. Math. 146* (2005), 357–370.

[6] —— Large groups, property (τ) and the homology growth of subgroups. (Preprint).

[7] LEVITT, G. On the cost of generating an equivalence relation. *Ergodic Theory Dynam. Systems 15* (1995), 1173–1181.

[8] LÜCK, W. *L^2-Invariants: Theory and Applications to Geometry and K-Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, *44*. Springer-Verlag, Berlin, 2002.

[9] MAGNUS, W., A. KARRASS and D. SOLITAR. *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations*. Interscience Publishers, John Wiley & Sons, Inc., New York-London-Sydney, 1966.

(Reçu le 4 septembre 2006)

Gábor Elek

The Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
P.O.B. 127
H-1364 Budapest
Hungary
e-mail: elek@renyi.hu