The irregularity of cyclic multiple planes after Zariski

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THE IRREGULARITY OF CYCLIC MULTIPLE PLANES AFTER ZARISKI

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ABSTRACT. A formula for the irregularity of a cyclic multiple plane associated to a branch curve that has arbitrary singularities and is transverse to the line at infinity is established. The irregularity is expressed as a sum of superabundances of linear systems associated to some multiplier ideals of the branch curve and the proof rests on the theory of standard cyclic coverings. Explicit computations of multiplier ideals are performed and some applications are presented.

1. INTRODUCTION

Let $f(x,y) = 0$ be an affine equation of a curve $B \subset \mathbf{P}^2$ and H_∞ be the line at infinity. The projective surface $S_0 \subset \mathbf{P}^3$ defined by the affine equation $z^n = f(x,y)$ is called by Zariski the *n-cyclic multiple plane* associated to B and H_∞ — possibly only to B if $n = \deg B$. For a given curve B, the cyclic multiple planes play an important role in the study of the fundamental group of the complement of B. At the same time they provide interesting examples of surfaces. In [23], Zariski took up the study of S_0 in the case that the curve B has only nodes and cusps and answered the following question: What is the *irregularity* of S_0, i.e. the dimension of the vector space of global holomorphic 1-forms on a desingularization of S_0 ?

ZARISKI'S THEOREM. *Let B be an irreducible curve of degree b, transverse to the line at infinity H_∞ and with only nodes and cusps as singularities. Let $S_0 \subset \mathbf{P}^3$ be the n-cyclic multiple plane associated to B and H_∞, and let S be a desingularization of S_0. The surface S is irregular if and only if n and b are both divisible by 6 and the linear system of curves of degree $5b/6-3$ passing through the cusps of B is superabundant. In this case,*

$$q(S) = h^1\Big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}\Big(-3+\frac{5b}{6}\Big)\Big),$$

where $\mathcal{Z}$ is the support of the set of cusps.

The aim of this paper is to present a generalization of Zariski's theorem to a branch curve that has arbitrary singularities and is transverse to the line at infinity bringing to the fore the theory of cyclic coverings as developed in [20]. The irregularity will be expressed as a sum of superabundances of linear systems defined in terms of some multiplier ideals associated to the branch curve B. We refer to [5] for the notion of multiplier ideal. To state the main result in Section 3, we recall here that if the rational ξ varies from a very small positive value to 1, then one can attach to B a collection of multiplier ideals $\mathcal{J}(\xi\cdot B)$ that starts at $\mathcal{O}_{\mathbf{P}^2}$, diminishes exactly when ξ equals a *jumping number* — they represent an increasing discrete sequence of rationals — and finally ends at $\mathcal{I}_B = \mathcal{O}_{\mathbf{P}^2}(-B)$. The multiplier ideals reflect the singularities of the rational curve ξB. For example in case B has only nodes and cusps, the only jumping number < 1 of B is $5/6$ and the corresponding multiplier ideal is $\mathcal{I}_{\mathcal{Z}}$, where $\mathcal{Z}$ is the support of the cusps.

THEOREM (3.1). *Let B be a plane curve of degree b and let H_∞ be a line transverse to B. Let S be a desingularization of the n-cyclic multiple plane associated to B and H_∞. If $J(B,n)$ is the subset of jumping numbers of B smaller than 1 and that live in $\frac{1}{\gcd(b,n)}\mathbf{Z}$, then*

$$q(S) = \sum_{\xi\in J(B,n)} h^1\big(\mathbf{P}^2, \mathcal{I}_{Z(\xi B)}(-3+\xi b)\big),$$

where $Z(\xi B)$ is the subscheme defined by the multiplier ideal $\mathcal{J}(\xi\cdot B)$.

Since for B with nodes and cusps $5/6$ is the only jumping number < 1, Theorem 3.1 becomes Zariski's theorem. In general, the usefulness of Theorem 3.1 relies on explicit computations of the jumping numbers and multiplier ideals attached to B. In case the singularities of B are locally given by equations of the form $x^p + y^q = 0$ such explicit computations may be performed and will enable us to apply the theorem to various examples in Section 4. Furthermore, in Remark 4.7 it will be shown that the irregularity may jump in case the position of H_∞ with respect to B becomes special.

Generalizations of Zariski's theorem are discussed in several papers and the proofs are based on different points of view. First, Zariski's original argument divides naturally into three parts. He describes the canonical system of S in terms of the conditions imposed by the singularities of S_0 that correspond to the cusps. Then he establishes the formula

$$q(S) = \sum_{k=n-\lfloor n/6 \rfloor}^{n-1} h^1\Big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}\Big(-3 + \Big\lceil \frac{kb}{n} \Big\rceil\Big)\Big), \tag{1.1}$$

where $\mathcal{Z}$ denotes the support of the set of cusps. To finish, he invokes the topological result proved in [22]: *If n is a power of a prime and B is irreducible, then the n-cyclic multiple plane is regular.* The theorem follows from the examination of the different terms in the previous sum when the degree of the cyclic multiple plane is a power of a prime and goes to infinity; these terms are

$$h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}(-3+5b/6+1)\big), h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}(-3+5b/6+2)\big), \ldots, h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}(-3+b)\big)$$

and they all vanish.

Second, Esnault establishes in [7] a formula similar to (1.1) for the irregularity of the b-cyclic multiple plane S_0, where b is the degree of the branch curve B that possesses arbitrary isolated singularities. She uses the techniques of logarithmic differential complexes, the existence of a mixed Hodge structure on the complex cohomology of the associated Milnor fibre — the complement of S_0 with respect to the plane that contains B — and the Kawamata-Viehweg vanishing theorem. In [1], Artal-Bartolo interprets Esnault's formula for the irregularity and applies it to produce two new Zariski pairs. Two plane curves $B_1, B_2 \subset \mathbf{P}^2$ are called a *Zariski pair* if they have the same degree and homeomorphic tubular neighbourhoods in $\mathbf{P}^2$, but the pairs $(\mathbf{P}^2, B_1)$ and $(\mathbf{P}^2, B_2)$ are not homeomorphic. Zariski was the first to discover that there are two types of plane sextics with six cuspidal singularities depending on whether or not the cusps lie on a plane conic. In [21], Vaquié gives a formula for the irregularity of a cyclic covering of degree n of a non-singular algebraic surface X ramified along a reduced curve B of degree b with respect to some projective embedding and a non-singular hyperplane section H that intersects B transversely. His formula is stated in terms of superabundances of the set of singularities of B and the proof also uses the techniques of logarithmic differential complexes. The superabundances involved are given by ideal sheaves that coincide in fact to the multiplier ideals. Vaquié's paper is one among several to introduce the notion of multiplier ideals implicitly and we refer to [5] for this issue.

Third, in [13], Libgober applies methods from knot theory to study the n-multiple plane S_0. His results are expressed in terms of Alexander polynomials and extend Zariski's theorem to irreducible curves B with arbitrary singularities and to lines H_∞ with arbitrary position with respect to B. Later on, in [14, 15, 16], he deals with the case of reducible curves B having transverse intersection with the line at infinity and the irregularity of the multiple plane is expressed using quasiadjunction ideals. The technique is based on mixed Hodge theory, and the result is a particular case in a vaster study pursued in the above mentioned papers where the homotopy groups of the complements of various divisors in smooth projective varieties are explored. These groups are related to the Hodge numbers of cyclic or more generally abelian coverings ramified along the considered divisors, as well as to the position of their singularities. We refer the reader to [18] for more ample details and references and to [17] for the relation between the quasiadjunction ideals and the multiplier ideals.

Our argument will follow Zariski's ideas. A desingularization of cyclic multiple plane is expressed as a standard cyclic covering. Then an analog of the formula (1.1) is obtained thanks to the theory of cyclic coverings:

$$q(S) = \sum_{k=1}^{n-1} h^1\left(\mathbf{P}^2, \mathcal{I}_{Z(\frac{k}{n}B)}\left(-3 + \left\lceil \frac{kb}{n} \right\rceil\right)\right).$$

Finally Theorem 3.1 is established using the Kawamata-Viehweg-Nadel vanishing theorem.

REMARK. The above formula coincides with Vaquié's in [21] when the latter is interpreted for a plane curve B and a line H transverse to it. At the same time, Vaquié's formula in its general form might be obtained by the argument we make use of in establishing Theorem 3.1 if Vaquié's general setting were to be considered.

The paper is organized as follows. In §2 the theory of cyclic coverings and some facts about multiplier ideals are recalled. Next, in §3 it is shown that the normalization of a given cyclic multiple plane is birationally isomorphic to a standard cyclic covering of the plane. Then, using it, Theorem 3.1 is proved. In §4 some applications are presented. Finally, in the appendix a new explicit computation for certain multiplier ideals is performed and used to complete the proof of Proposition 4.3. It is hoped that this description might be useful in other circumstances.

NOTATION AND CONVENTIONS. All varieties are assumed to be defined over $\mathbf{C}$. Standard symbols and notation in algebraic geometry will be freely used. The multiplier ideal associated to a curve B and a rational ξ will be denoted by $\mathcal{J}(\xi \cdot B)$ and the corresponding subscheme by $Z(\xi B) = Z(\mathcal{J}(\xi \cdot B))$. If Z is a subscheme in X, then $\mathcal{I}_Z$ is the sheaf of ideals locally defined by the functions that vanish along Z. In particular, $\mathcal{J}(\xi \cdot B) = \mathcal{I}_{Z(\xi B)}$. Moreover, if D is a divisor on the variety Y, we shall often write $H^i(Y, D)$ and $h^i(Y, D)$ instead of $H^i(Y, \mathcal{O}_Y(D))$ and $h^i(Y, \mathcal{O}_Y(D))$ respectively. If $\mathcal{L}$ is an invertible sheaf on Y, then we shall regularly denote by L a divisor such that $\mathcal{L} \simeq \mathcal{O}_Y(L)$.

For m a positive integer, if $a \in \mathbf{Z}/m$ then $a^{\bullet}$ will denote the smallest non-negative integer in the equivalence class a.

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2. PRELIMINARIES

We shall summarize, in a form convenient for further use, some properties of cyclic coverings and of multiplier ideals.

2.1 CYCLIC COVERINGS

Let Y be a variety and let G be the cyclic group of order n. If G acts faithfully on Y, then the quotient $X = Y/G$ exists and Y is called an *abelian covering* of X with group G. The map $\pi \colon Y \to X$ is a finite morphism, $\pi_* \mathcal{O}_Y$ is a coherent sheaf of $\mathcal{O}_X$-algebras, and $Y \simeq \mathbf{Spec}_{\mathcal{O}_X}(\pi_* \mathcal{O}_Y)$.

If Y is normal and X is smooth, then π is flat and consequently $\pi_*\mathcal{O}_Y$ is locally free of rank n. The action of G on $\pi_*\mathcal{O}_Y$ decomposes it into the direct sum of eigen line bundles associated to the characters $\chi \in \widehat{G} = \mathrm{Hom}(G, \mathbf{S}^1)$,

$$\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \bigoplus_{\chi\in\widehat{G},\chi\neq 1} \mathcal{L}_\chi^{-1}.$$

The action of G on $\mathcal{L}_\chi^{-1}$ is the multiplication by χ.

There are two naturally arising questions when dealing with cyclic coverings. First, what is the ring structure of $\pi_*\mathcal{O}_Y$? Knowing this structure is equivalent to knowing the covering Y. This structure, being compatible with the group action, is determined by the multiplications $\mathcal{L}_\chi^{-1} \otimes \mathcal{L}_{\chi'}^{-1} \to \mathcal{L}_{\chi\chi'}^{-1}$ for any $\chi, \chi' \in \widehat{G}$. Finding the image of each of these maps will provide us with an answer to the first question and lead us to ask the second one: Given a covering Y of X, is there straightforward information at the level of X — less involved than $n-1$ line bundles $\mathcal{L}_\chi$, $\chi \in \widehat{G}, \chi \neq 1$, and a ring structure on $\bigoplus_{\chi\in\widehat{G}} \mathcal{L}_\chi^{-1}$ — for telling us how to reconstruct Y ?

EXAMPLE 2.1 (Simple coverings). Let $B \subset X$ be a reduced effective divisor such that there exists a line bundle $\mathcal{L}$ over X with $\mathcal{L}^n \simeq \mathcal{O}_X(B)$. This data (later on it will be called *reduced building data*) defines an n-cyclic covering of X totally ramified along B: let ψ be a fixed generator for $\widehat{G}$, let $\mathcal{L}_{\psi^k} = \mathcal{L}^k$ and let

$$\mathcal{L}_{\psi^j}^{-1} \otimes \mathcal{L}_{\psi^k}^{-1} \xrightarrow{\simeq} \mathcal{L}_{\psi^{j+k}}^{-1} \otimes \mathcal{O}_X(-\varepsilon(j,k)B) \hookrightarrow \mathcal{L}_{\psi^{j+k}}^{-1}$$

be the multiplications for any $1 \le j, k \le n-1$, with $\varepsilon(j,k) = 0$ or 1 depending on whether or not $j+k < n$. If $L \xrightarrow{p} X$ denotes the total space of $\mathcal{L}$ with z the tautological section of $p^*\mathcal{L}$, then Y is defined in L by $z^n - p^*s = 0$, where s is a global section defining B.

Before turning to the two questions asked formerly, let us notice that a general cyclic covering Y may be seen as a subvariety into a vector bundle over X in the same way a simple covering was seen into a line bundle. Let $\mathcal{F} = \bigoplus_{\chi\neq 1} \mathcal{L}_\chi^{-1}$. The surjection $\mathrm{Sym}_{\mathcal{O}_X} \mathcal{F} \to \pi_*\mathcal{O}_Y$ defines the embedding of Y into the total space of $\mathcal{F}$, $F \xrightarrow{p} X$. The ring structure of $\pi_*\mathcal{O}_Y$ is equivalent to knowing the kernel of that surjection. Over an open subset $U \subset X$, if z_j denotes the tautological section of the line bundle $p^*\mathcal{L}_{\chi^j}$, the surjection $\mathrm{Sym}_{\mathcal{O}_X} \mathcal{F} \to \pi_*\mathcal{O}_Y$ becomes

$$\mathcal{O}_X(U)[z_1, \ldots, z_{n-1}] \longrightarrow (\pi_*\mathcal{O}_Y)(U) = \mathcal{O}_Y(\pi^{-1}(U)). \tag{2.1}$$

To understand the ring structure of $\pi_*\mathcal{O}_Y$ let us consider a component D of the ramification locus. We suppose that Y is normal and X is smooth. Since π is flat, D is 1-codimensional. The component D is associated to its *inertia subgroup* $H \subset G$ — the subset of elements of G that globally fix D — and to a character $\psi \in \widehat{H}$ that generates $\widehat{H}$. The character ψ corresponds to the induced representation of H on the cotangent space to Y at D. Dualizing the inclusion $H \subset G$, such a couple (H, ψ) is equivalent to a group epimorphism $f\colon \widehat{G} \to \mathbf{Z}/m_f$, where $m_f = |H|$; for any $\chi \in \widehat{G}$, the induced representation $\chi|_H$ is given by $\psi^{f(\chi)^\bullet}$.

Recall that $a^\bullet$ denotes the smallest non-negative integer in the equivalence class of $a \in \mathbf{Z}/m$. Here and later on, $\mathfrak{F}$ denotes the set of all group epimorphisms from $\widehat{G}$ to different $\mathbf{Z}/m\mathbf{Z}$. Let $B_f \subset X$ be the subdivisor of the branch locus defined set-theoretically as $\pi(R_f)$, with R_f the union of all the components D of the ramification locus associated to the group epimorphism f. In [20] it is shown that the ring structure is given by the following isomorphisms: for any $\chi, \chi' \in \widehat{G}$,

$$\mathcal{L}_\chi \otimes \mathcal{L}_{\chi'} \simeq \mathcal{L}_{\chi\chi'} \otimes \bigotimes_{f \in \mathfrak{F}} \mathcal{O}_X(\varepsilon(f, \chi, \chi')B_f) \tag{2.2}$$

with $\varepsilon(f, \chi, \chi') = 0$ or 1, depending on whether or not $f(\chi)^\bullet + f(\chi')^\bullet < m_f$.

EXAMPLE 2.2. Let P and Q be two distinct points of $\mathbf{P}^1$. We define $\mathcal{L}_\chi = \mathcal{L}_{\chi^2} = \mathcal{O}_{\mathbf{P}^1}(1)$ and a ring structure on $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{L}_\chi^{-1} \oplus \mathcal{L}_{\chi^2}^{-1}$ by the isomorphisms

$$\mathcal{L}_\chi \otimes \mathcal{L}_\chi \simeq \mathcal{L}_{\chi^2} \otimes \mathcal{O}_{\mathbf{P}^1}(Q), \quad \mathcal{L}_\chi \otimes \mathcal{L}_{\chi^2} \simeq \mathcal{O}_{\mathbf{P}^1} \otimes \mathcal{O}_{\mathbf{P}^1}(P+Q)$$
$$\text{and } \mathcal{L}_{\chi^2} \otimes \mathcal{L}_{\chi^2} \simeq \mathcal{L}_\chi \otimes \mathcal{O}_{\mathbf{P}^1}(P).$$

We obtain the triple covering Y of $\mathbf{P}^1$ totally ramified over P and Q. For example above $\mathbf{P}^1 - \{Q\}$, if x is a local coordinate centered at P, Y is defined by the surjection

$$\mathbf{C}[x][z_1, z_2] \longrightarrow \mathbf{C}[x][z_1, z_2]/(z_1^2 - z_2, z_1z_2 - x, z_2^2 - xz_1) \simeq \mathbf{C}[x][z_1]/(z_1^3 - x).$$

Similarly, above $\mathbf{P}^1 - \{P\}$ with y the local coordinate $y(Q) = 0$, the triple covering is defined by $\mathbf{C}[y][z_2]/(z_2^3 - y)$. In other words, locally Y looks like a simple triple covering, but globally it is not a simple covering.

The next proposition is formulated for cyclic groups, since it is this case that will be used in the sequel. We refer again to [20] for the case of abelian groups.

PROPOSITION 2.3. *Let $\pi\colon Y \to X$ be a cyclic covering with Y normal and X smooth. If ψ generates $\widehat{G}$, then for every $k = 1, \dots, n$,*

$$L_{\psi^k} \sim kL_\psi - \sum_{f\in\mathfrak{F}} \left\lfloor \frac{kf(\psi)^\bullet}{m_f} \right\rfloor B_f . \tag{2.3}$$

In particular, for $k = n$ equation (2.3) becomes

$$nL_\psi \sim \sum_{f\in\mathfrak{F}} \frac{n}{m_f} f(\psi)^\bullet B_f . \tag{2.4}$$

Proof. From the hypothesis, ψ spans the group of characters. Applying (2.2) for ψ and ψ^{j-1} we get

$$L_\psi + L_{\psi^{j-1}} \sim L_{\psi^j} + \sum_{f\in\mathfrak{F}} \varepsilon(f, \psi, \psi^{j-1}) B_f .$$

Then, summing over j from 1 to k,

$$L_{\psi^k} \sim kL_\psi - \sum_{j=1}^{k} \sum_{f\in\mathfrak{F}} \varepsilon(f, \psi, \psi^{j-1}) B_f = kL_\psi - \sum_{f\in\mathfrak{F}} \sum_{j=1}^{k} \varepsilon(f, \psi, \psi^{j-1}) B_f .$$

By definition $\varepsilon(f, \psi, \psi^{j-1}) = 1$ is equivalent to $f(\psi)^\bullet + f(\psi^{j-1})^\bullet \geq m_f$ which is equivalent to $(f(\psi) + f(\psi^{j-1}))^\bullet < f(\psi)^\bullet$, i.e. to $(jf(\psi))^\bullet < f(\psi)^\bullet$. It follows that $\sum_{j=1}^{k} \varepsilon(f, \psi, \psi^{j-1})$ counts the number of j's in $\{1, 2, \dots, k\}$ for which $(jf(\psi))^\bullet < f(\psi)^\bullet$, i.e. for which the remainder of the division of $jf(\psi)$ — equivalently of $jf(\psi)^\bullet$ — by m_f is smaller than $f(\psi)^\bullet$. This number is exactly $\lfloor kf(\psi)^\bullet / m_f \rfloor$ and formula (2.3) follows. Formula (2.4) is obvious, since $\psi^n = 1$. $\square$

We are now able to answer the second question. Starting with a line bundle $\mathcal{L}_\psi$, a fixed generator ψ of $\widehat{G}$, and effective divisors B_f, $f \in \mathfrak{F}$, that satisfy the identity (2.4), we define the line bundles $\mathcal{L}_{\psi^k}$ using formula (2.3). Any three of these line bundles $\mathcal{L}_\chi, \mathcal{L}_{\chi'}$ and $\mathcal{L}_{\chi\chi'}$ verify equation (2.2). Consequently, the $\mathcal{O}_X$-module $\bigoplus_k \mathcal{L}_{\psi^k}^{-1}$ is endowed with a ring structure, hence it defines in a natural way *the standard cyclic covering* $\pi\colon Y = \mathbf{Spec}_{\mathcal{O}_X}(\bigoplus_k \mathcal{L}_{\psi^k}^{-1}) \to X$. In case Y is normal the covering is unique up to isomorphisms of cyclic coverings. We notice that when we started the investigation of the ring structure we supposed Y normal and denoted by B_f some components of the branch divisor defined set theoretically, hence without multiple components. Now in the construction of the standard cyclic covering the divisors B_f may have multiple components. For example starting with $B_f = P + 2Q$ on $\mathbf{P}^1$, the standard covering defined by $3L_\psi \sim B_f$ is the simple 3-covering (see Example 2.1) ramified above P and Q and having a cuspidal point over Q.

Following [20], we will call the divisors $\mathcal{L}_\chi$ and B_f, $f \in \mathfrak{F}$, used in the definition of a standard cyclic covering a set of *reduced building data* for the covering.

2.2 THE NORMALIZATION PROCEDURE FOR STANDARD CYCLIC COVERINGS

The standard covering obtained starting with a set of reduced building data may not be normal. In [20, Corollary 3.1] it is shown that such a standard covering is not normal precisely above the multiple components of the branch locus and the normalization procedure is constructed. Let $f\colon \widehat{G} \to \mathbf{Z}/m_f$ be a group epimorphism and let $B_f = rC + R$, with C irreducible, C not a component of R and $r \geq 2$. The surface Y is not normal along the pull-back of C. The normalization procedure along this multiple component splits into two steps and shows how to end up with a new covering, normal along the pull-back of C. We shall later review the formulae involved for each step. They are based on the comparison between the multiplicity r and the order m_f of the inertia subgroup. Two simple examples should shed some light on these steps.

EXAMPLE 2.4 (for the first step). Suppose that s is a coordinate along the affine line, that $Y \to \mathbf{A}^1$ is given by $z^m - s^d = 0$ in the affine plane and that d divides m. The curve $Y = \operatorname{Spec} \mathbf{C}[s,z]/(z^m - s^d)$ is a simple cyclic covering of the line ramified above the origin. It is smooth, or equivalently normal, if and only if $d = 1$. If $d > 1$, a desingularization Y' of Y is defined by the $\mathbf{C}[s]$-algebra $\mathbf{C}[s,z,\zeta]/(z^{m/d} - \zeta s, \zeta^d - 1)$. The inclusion of $\mathbf{C}[s]$-algebras

$$\mathbf{C}[s,z]/(z^m - s^d) \xhookrightarrow{i_1} \mathbf{C}[s,z,\zeta]/(\zeta^d - 1, z^{m/d} - \zeta s)$$

tells us that the covering $Y' \to \mathbf{A}^1$ factors through an étale covering of the affine line of degree d, $Y' \to Y_{et} \to \mathbf{A}^1$.

EXAMPLE 2.5 (for the second step). This time, suppose that $Y \to \mathbf{A}^1$ is the simple cyclic covering given by $z^m - s^r = 0$ in the affine plane and that m and $r \geq 2$ are relatively prime positive integers. Let the positive integers q and v satisfy $vr - qm = 1$. A desingularization Y' of Y is defined by the inclusion of $\mathbf{C}[s]$-algebras

$$\mathbf{C}[s,z]/(z^m - s^r) \xhookrightarrow{i_2} \mathbf{C}[s,\xi]/(\xi^m - s),$$

with $i_2(z) = \xi^r$. It says that the covering $Y \to \mathbf{A}^1$ is desingularized by the change of coordinates $\xi = z^v/s^q$ since we have $\xi^r = z^{vr}/s^{qr} = z$.

STEP 1. If $B_f = rC + R$ and $(r, m_f) = d > 1$, then the natural composition is considered

$$f' : \widehat{G} \xrightarrow{f} \mathbf{Z}/m_f \longrightarrow \mathbf{Z}/\frac{m_f}{d}\,.$$

For any χ, the integers $f(\chi)^{\bullet}$ and $f'(\chi)^{\bullet}$ are linked by the relation $f(\chi)^{\bullet} = q_\chi m_f/d + f'(\chi)^{\bullet}$. Put

$$L'_\chi \sim L_\chi - q_\chi \frac{r}{d} C\,, \quad B'_f = R\,, \quad B'_{f'} = B_{f'} + \frac{r}{d} C \quad \text{and} \quad B'_g = B_g \quad \text{if } g \neq f, f'$$

in order to construct $Y' \to X$, a 'less non-normal' covering over C.

Two facts should be noticed. Firstly, if $\psi \in \widehat{G}$ is such that $f(\psi) = 1$, then Y is a simple covering locally over $X \setminus \bigcup_{g \neq f} B_g$ defined by $\mathcal{L}_\psi$. The new covering $Y' \to X$ factors over the same open subset through an étale covering of X of degree d followed by a simple covering of degree m_f/d defined by the pull-back of $\mathcal{L}'_\psi$ on the étale covering. By Proposition 2.3, $L_{\psi^{m_f/d}} \sim m_f/d\, L_\psi$, then

$$L'_{\psi^{m_f/d}} \sim L_{\psi^{m_f/d}} - \frac{r}{d} C \sim \frac{m_f}{d} L_\psi - \frac{r}{d} C\,,$$

hence

$$dL'_{\psi^{m_f/d}} \sim 0 \qquad \text{and} \qquad \frac{m_f}{d} L'_\psi \sim \frac{m_f}{d} L_\psi \sim L'_{\psi^{m_f/d}} + \frac{r}{d} C\,.$$

These relations, seen in terms of tautological sections of the corresponding line bundles as in (2.1), are exactly the relations from Example 2.4. Secondly, looking at f', the induced multiplicity and the corresponding subgroup order become relatively prime.

STEP 2. If $B_f = rC + R$ with $r \geq 2$ and $(r, m_f) = 1$, the composition

$$f' : \widehat{G} \xrightarrow{f} \mathbf{Z}/m_f \xrightarrow{r \cdot} \mathbf{Z}/m_f$$

is considered. As before, for any $\chi \in \widehat{G}$, the integers $f(\chi)^{\bullet}$ and $f'(\chi)^{\bullet}$ are linked by $r \cdot f(\chi)^{\bullet} = q_\chi m_f + f'(\chi)^{\bullet}$. Put

$$L'_\chi \sim L_\chi - q_\chi C\,, \quad B'_f = R\,, \quad B'_{f'} = B_{f'} + C \quad \text{and} \quad B'_g = B_g \quad \text{if } g \neq f, f'$$

to get a new covering — if $f(\psi) = 1$ and $vr - qm = 1$, then over $X \setminus (R \cup \bigcup_{g \neq f} B_g)$ the covering Y is simple defined by $\mathcal{L}_\psi$, $\mathcal{L}_{\psi^v} \simeq \mathcal{L}_\psi^{\otimes v}$ and Y' is simple and defined by $\mathcal{L}'_{\psi^v}$ as in Example 2.5 — and finish the normalization procedure along C.

EXAMPLE 2.6. On $\mathbf{P}^2$ let $\mathcal{L}_\chi = \mathcal{O}(1)$ and $nL_\chi \sim H_0 + (n-1)H_\infty$, where H_0 and H_∞ are two different fixed lines. The simple cyclic n-covering $Y \to \mathbf{P}^2$ given by the set of reduced building data $\mathcal{L}_\chi$ and $B_f = H_0 + (n-1)H_\infty$, where $f(\chi) = 1$, is not normal above H_∞. Applying the second step of the normalization procedure, if $f'\colon \widehat{G} \to \mathbf{Z}/n\mathbf{Z}$ is defined by $f'(\chi) = n-1$, we obtain the normalization Y' of Y as the n-cyclic covering with building data $\mathcal{L}'_\chi = \mathcal{O}(1)$, $B_f = H_0$ and $B_{f'} = H_\infty$. Clearly Y' has a singular point above P, the intersection of H_0 and H_∞. Actually we may obtain a desingularization of Y using the theory of cyclic coverings. We consider the blow-up surface $\mathrm{Bl}_P\mathbf{P}^2$, with E the exceptional divisor and the induced simple cyclic covering $S \to \mathrm{Bl}_P\mathbf{P}^2$ with building data $L_\chi = \mathcal{O}_{\mathrm{Bl}_P\mathbf{P}^2}(H)$ and $B_f = H_0 + (n-1)H_\infty + nE$. Curves on $\mathbf{P}^2$ and their strict transforms are denoted by the same symbol. This time the normalization procedure requires the first and the second step and leads to $S' \to \mathrm{Bl}_P\mathbf{P}^2$ defined by $nL'_\chi \sim H_0 + (n-1)H_\infty$, with $\mathcal{L}'_\chi = \mathcal{O}_{\mathrm{Bl}_P\mathbf{P}^2}(H-E)$, $B_f = H_0$ and $B_{f'} = H_\infty$. Incidentally, the surface S' may be identified. The lines in the plane through P tell us that S' is a geometrically ruled surface. Besides, the pull-back of E is a rational section with self-intersection $-n$, hence S' is the Hirzebruch surface $\boldsymbol{F}_n$.

EXAMPLE 2.7. On $\mathbf{P}^2$ let B be a reduced curve of degree b, H_∞ a fixed line and $n \geq 2$ a fixed integer. For an integer $r \geq 0$, the identity $nL_\psi \sim B + rH_\infty$ defines a simple n-covering $S_r \to \mathbf{P}^2$ if and only if n divides $r+b$. A set of reduced building data for the covering is represented by $\mathcal{L}_\psi \sim \mathcal{O}_{\mathbf{P}^2}((r+b)/n)$ and $B_f = B + rH_\infty$, with $f(\psi) = 1$. If $r > 1$, the normalization procedure leads to the standard cyclic covering S' which is independent of r. It is defined by $\mathcal{L}'_\psi = \mathcal{O}_{\mathbf{P}^2}(\lceil b/n \rceil)$, $B'_f = B$ and $B_g = H_\infty$, where $f\colon \widehat{G} \to \mathbf{Z}/n$, $f(\psi) = 1$, and

$$g\colon \widehat{G} \to \mathbf{Z}/\frac{n}{\gcd(n, \lceil b/n \rceil n - b)}, \qquad g(\psi) = \frac{\lceil b/n \rceil n - b}{\gcd(n, \lceil b/n \rceil n - b)}.$$

We shall justify the assertion when the integers n and r are relatively prime and leave the more involved case as an exercise. The normalization procedure is reduced to the second step and $g\colon \widehat{G} \to \mathbf{Z}/n$ is the composition of f with the multiplication by r in $\mathbf{Z}/n$. Then

$$g(\psi)^\bullet = r - \left\lfloor \frac{r}{n} \right\rfloor n = \frac{r+b}{n} n - b - \left\lfloor \frac{r}{n} \right\rfloor n = \left\lceil \frac{b}{n} \right\rceil n - b$$

and

$$\mathcal{L}'_\psi = \mathcal{L}_\psi \otimes \mathcal{O}_{\mathbf{P}^2}\left(-\left\lfloor \frac{r}{n} \right\rfloor\right) \simeq \mathcal{O}_{\mathbf{P}^2}\left(\frac{b+r}{n}\right) \otimes \mathcal{O}_{\mathbf{P}^2}\left(-\left\lfloor \frac{r}{n} \right\rfloor\right) = \mathcal{O}_{\mathbf{P}^2}\left(\left\lceil \frac{b}{n} \right\rceil\right).$$

2.3 Multiplier ideals

In this subsection we briefly recall the notion of multiplier ideal of divisors, the other foremost tool of the paper. We refer the reader to [12] for the many contexts where multiplier ideals appear and for the results that are cited below.

Let X be a smooth variety, $D \subset X$ be an effective $\mathbf{Q}$-divisor and $\mu: Y \to X$ be a *log resolution* for D, i.e. the support of the $\mathbf{Q}$-divisor $K_{Y|X} - \mu^*D$ is a union of irreducible smooth divisors with normal crossing intersections. Then $\mu_*\mathcal{O}_Y(K_{Y|X} - \lfloor \mu^*D \rfloor)$ is an ideal sheaf $\mathcal{J}(D)$ on X. We will denote by $Z(D)$ the subscheme defined by this ideal. Hence $\mathcal{I}_{Z(D)} = \mathcal{J}(D)$. Showing that $\mathcal{J}(D)$ is independent of the choice of the resolution, we have:

Definition. The ideal $\mathcal{J}(D) = \mu_*\mathcal{O}_Y(K_{Y|X} - \lfloor \mu^*D \rfloor)$ is called the *multiplier ideal* of D.

Examples 2.8. 1) Let X be a smooth surface and $B \subset X$ a smooth curve except at the point P where B has a simple double point — a node. Then for any rational $0 < \xi < 1$ we have that

$$\mathcal{J}(\xi \cdot B) = \mu_*\mathcal{O}_Y(K_{Y|X} - \lfloor \mu^*\xi \cdot B \rfloor) = \mu_*\mathcal{O}_Y(E - \lfloor 2\xi \rfloor E) = \mathcal{O}_X,$$

since the blow-up of X at P is a log resolution for B and $\mu_*\mathcal{O}_Y(E) = \mu_*\mathcal{O}_Y = \mathcal{O}_X$.

2) We keep the same notation, but suppose that the singularity of B at P is a simple triple point, i.e. in local coordinates it is given by $x^3 + y^3 = 0$. Then $\mathcal{J}(\xi \cdot B) = \mu_*\mathcal{O}_Y(E - \lfloor 3\xi \rfloor E)$, so $\mathcal{J}(\xi \cdot B) = 0$ for any $0 < \xi < 2/3$ and $\mathcal{J}(\xi \cdot B) = \mathcal{I}_P$ for any $2/3 \leq \xi < 1$.

The sheaf computing the multiplier ideal verifies the following local vanishing result: for every $i > 0$, $R^i\mu_*\mathcal{O}_Y(K_{Y|X} - \lfloor \mu^*D \rfloor) = 0$. Therefore, applying the Leray spectral sequence, we obtain that for every i and any Cartier divisor L on X,

$$(2.5)\quad H^i\big(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_{Z(D)}\big) = H^i\big(Y, \mathcal{O}_Y(\mu^*K_X + \mu^*L + K_{Y|X} - \lfloor \mu^*D \rfloor)\big).$$

In the example below we consider a simple instance of how the multiplier ideals appear in the computation of the irregularity of multiple planes.

Example 2.9. Let L_1, L_2 and L_3 be three lines in the plane that intersect in P and let S_0 be the simple cyclic 3-covering given by the line bundle $\mathcal{O}_{\mathbf{P}^2}(1)$ and by $B = B_f = L_1 + L_2 + L_3$ with $f(\psi) = 1$. After blowing up the plane at P and normalizing the induced triple covering, we obtain the desingularization of S_0, a smooth simple 3-covering $S \xrightarrow{\pi} \mathrm{Bl}_P\,\mathbf{P}^2$ given by $\mathcal{L}_\psi = \mathcal{O}_{\mathrm{Bl}_P\,\mathbf{P}^2}(H-E)$ and ramified over the strict transforms of the lines L_j. The exceptional divisor has been denoted by E. The covering being simple, the canonical divisor of S is $K_S = \pi^*(K_{\mathrm{Bl}_P\,\mathbf{P}^2} + 2L_\psi)$. We have

$$h^1\big(S, \pi^*(K_{\mathrm{Bl}_P\,\mathbf{P}^2} + 2L_\psi)\big) = h^1(\mathrm{Bl}_P\,\mathbf{P}^2, K_{\mathrm{Bl}_P\,\mathbf{P}^2} + 2L_\psi) = h^1(\mathrm{Bl}_P\,\mathbf{P}^2, -H-E)\,,$$

hence $q(S) = h^1\big(\mathbf{P}^2, \mathcal{I}_P(-1)\big)$. To see how the notion of multiplier ideal appears in this computation, in fact how $\mathcal{I}_P$ is naturally seen as $\mathcal{J}(\frac{2}{3}\cdot B)$, notice that $\mu\colon \mathrm{Bl}_P\,\mathbf{P}^2 \to \mathbf{P}^2$ is a log resolution for the divisor $B = L_1+L_2+L_3$ at the triple point and that $2L_\psi = 2(L_1+L_2+L_3)/3 = 2H - \lfloor \mu^*2B/3 \rfloor$. We have

$$K_{\mathrm{Bl}_P\,\mathbf{P}^2} + 2L_\psi \sim \mu^* K_{\mathbf{P}^2} + 2H + \Big(K_{\mathrm{Bl}_P\,\mathbf{P}^2|\mathbf{P}^2} - \Big\lfloor \mu^* \frac{2}{3} B \Big\rfloor\Big)$$

and using (2.5),

$$q(S) = h^1(S, K_S) = h^1\big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-1) \otimes \mathcal{J}(2/3\cdot B)\big)\,.$$

For multiplier ideals, the basic global vanishing theorem is the following:

KAWAMATA-VIEHWEG-NADEL VANISHING THEOREM. *Let X be a smooth projective variety. If L is a Cartier divisor and D is an effective $\mathbf{Q}$-divisor on X such that $L-D$ is a nef and big $\mathbf{Q}$-divisor, then*

$$h^i\big(X, \mathcal{O}_X(K_X+L) \otimes \mathcal{I}_{Z(D)}\big) = 0$$

for every $i > 0$.

DEFINITION-LEMMA (see [5]). Let $B \subset X$ be an effective divisor and $P \in B$ be a fixed point. Then there is an increasing discrete sequence of rational numbers $\xi_i = \xi(B,P)$,

$$0 = \xi_0 < \xi_1 < \dots$$

such that

$$\mathcal{J}(\xi B)_P = \mathcal{J}(\xi_i B)_P \quad \text{for every} \quad \xi \in [\xi_i, \xi_{i+1})\,,$$

and $\mathcal{J}(\xi_{i+1}B)_P \subsetneq \mathcal{J}(\xi_i B)_P$. The rational numbers ξ_i's are called *the jumping numbers* of B at P.

3. THE IRREGULARITY OF CYCLIC MULTIPLE PLANES

THEOREM 3.1. *Let B be a plane curve of degree b and let H_∞ be a line transverse to B. Let S be a desingularization of the projective n-cyclic multiple plane associated to B and H_∞. If*

$$J(B,n) = \left\{\xi \mid \xi \text{ jumping number of } B,\ 0<\xi<1,\ \xi \in \frac{1}{\gcd(b,n)}\mathbf{Z}\right\},$$

then

$$q(S) = \sum_{\xi \in J(B,n)} h^1\big(\mathbf{P}^2, \mathcal{I}_{Z(\xi B)}(-3+\xi b)\big),$$

with $Z(\xi B)$ the subscheme defined by the multiplier ideal $\mathcal{J}(\xi \cdot B)$.

Proof. To compute the irregularity of a desingularization of S_0 we need either to desingularize S_0, or to find a smooth surface birationally equivalent to S_0. We shall follow the latter possibility. Let $S_1 \to \mathbf{P}^2$ be the normal standard covering defined by the reduced building data $\mathcal{L}'_\psi = \mathcal{O}_{\mathbf{P}^2}(\lceil b/n \rceil), B_f = B$ and $B_g = H_\infty$, where ψ is a generator of $\widehat{G}$, $f\colon \widehat{G} \to \mathbf{Z}/n$, $f(\psi)=1$, and

$$g\colon \widehat{G} \to \mathbf{Z}/\frac{n}{\gcd(n, \lceil b/n\rceil\, n - b)}, \qquad g(\psi) = \frac{\lceil b/n\rceil\, n - b}{\gcd(n, \lceil b/n\rceil\, n-b)}.$$

It might be noticed that by Example 2.7 the surface S_1 is the normalization of any n-standard covering of the plane ramified along B and along a multiple of the line at infinity. The relation defining S_1 is

$$nL'_\psi \sim B + (\lceil b/n\rceil\, n - b)H_\infty .$$

Over $\mathbf{A}^2 = \mathbf{P}^2 \smallsetminus H_\infty$ the covering S_1 coincides with the affine surface Σ defined by $z^n = f(x,y)$, with $f(x,y)=0$ an equation for $B \smallsetminus H_\infty \subset \mathbf{A}^2$. The surfaces S_1 and the normalization S'_0 of S_0 are birationally equivalent. In fact, since they are normal and $S_1 \to \mathbf{P}^2$ is finite, $S'_0 \to S_1$ is a birational morphism.

We compute the irregularity of the multiple plane S_0 using the standard covering S_1. If $\mu\colon X \to \mathbf{P}^2$ is a desingularization of B such that its total transform on X is a divisor with normal crossing intersections, i.e. if μ is a log resolution for B, then the standard cyclic covering S_1 pulls back to a standard cyclic covering S_2 of X. The normalization procedure yields a normal surface S with only Hirzebruch-Jung singularities (see [20], Proposition 3.3). We have the diagram shown in Figure 1.

If $\mathcal{L}_\psi$ denotes the line bundle defining S, we need to control the line bundles $\mathcal{L}_{\psi^k}$ in order to express the irregularity of a desingularization of S as a sum of some h^1's. The proof will be concluded by applying the Kawamata-Viehweg-Nadel vanishing theorem. We need two preliminary results.

$$\begin{tikzcd} S \arrow[dr] \arrow[ddr, "\pi"'] & & S_0' \arrow[r] \arrow[d] & S_0 \\ & S_2 \arrow[r] \arrow[d] & S_1 \arrow[d] & \Sigma \arrow[l] \arrow[u] \arrow[d] \\ & X \arrow[r, "\mu"] & \mathbf{P}^2 & \mathbf{A}^2 \arrow[l] \end{tikzcd}$$

FIGURE 1

PROPOSITION 3.2. *Let X be smooth and let $\pi: Y \to X$ be a standard cyclic covering of degree n determined by the set of reduced building data $\mathcal{L}_\psi$ and B_f, $f \in \mathfrak{F}$, i.e. by $nL_\psi \sim \sum_{f\in\mathfrak{F}} n/m_f f(\psi)^\bullet B_f$. For a fixed $g \in \mathfrak{F}$, the branching divisor B_g is supposed to have a multiple component, say $B_g = rC + R$ with $r > 1$. Let $Y'' \to X$ be the standard cyclic covering obtained from Y after the normalization procedure has been applied to the multiple component rC. If Y'' is associated to*

$$nL''_\psi \sim \sum_{f\in\mathfrak{F}} \frac{n}{m_f} f(\psi)^\bullet B''_f \,,$$

then for every $k = 1, \ldots, n-1$,

$$L''_{\psi^k} \sim kL_\psi - \left\lfloor \frac{krg(\psi)^\bullet}{m_g} \right\rfloor C - \left\lfloor \frac{kg(\psi)^\bullet}{m_g} \right\rfloor R - \sum_{f\neq g} \left\lfloor \frac{kf(\psi)^\bullet}{m_f} \right\rfloor B_f \,.$$

Proof. We present the proof in case both steps of the normalization procedure from the Subsection 2.2 are needed. Otherwise the argument is easier. So suppose that $(r, m_g) = d > 1$ and consider the map

$$g' : \widehat{G} \xrightarrow{g} \mathbf{Z}/m_g \to \mathbf{Z}/\frac{m_g}{d} \,.$$

For any $\chi \in \widehat{G}$ the integer $g(\chi)^\bullet$ satisfies

$$g(\chi)^\bullet = q_\chi \frac{m_g}{d} + g'(\chi)^\bullet \,. \tag{3.1}$$

The covering data are modified to

$$L'_\chi \sim L_\chi - q_\chi \frac{r}{d} C \,, \quad B'_g = R \,, \quad B'_{g'} = B_{g'} + \frac{r}{d} C \,, \tag{3.2}$$
$$B'_f = B_f \quad \text{for} \quad f \neq g, g' \,.$$

Now the multiplicity r/d of C is an integer greater than 1 and prime to m_g/d. Consider the map $g'' : \widehat{G} \xrightarrow{g'} \mathbf{Z}/\frac{m_g}{d} \xrightarrow{r/d \cdot} \mathbf{Z}/\frac{m_g}{d}$. We have

$$\frac{r}{d}\,g'(\chi)^{\bullet} = q'_{\chi}\frac{m_g}{d} + g''(\chi)^{\bullet} \tag{3.3}$$

and the covering data are modified to

$$\begin{gathered} L''_{\chi} \sim L'_{\chi} - q'_{\chi}C\,, \quad B''_{g'} = B'_{g'}\,, \quad B''_{g''} = B'_{g''} + C\,, \\ B''_f = B'_f \quad \text{for} \quad f \neq g', g''\,. \end{gathered} \tag{3.4}$$

Using (3.2) and (3.4) we have $L''_{\chi} \sim L_{\chi} - (q_{\chi}r/d + q'_{\chi})C$ for any $\chi \in \widehat{G}$. By Proposition 2.3, $L''_{\psi^k} \sim kL''_{\psi} - \sum_f \lfloor kf(\psi)^{\bullet}/m_f \rfloor B''_f$ for any $k = 0, \ldots, n-1$, so L''_{ψ^k} is linearly equivalent to

$$kL''_{\psi} - \left\lfloor \frac{kg(\psi)^{\bullet}}{m_g} \right\rfloor R - \left\lfloor \frac{kg'(\psi)^{\bullet}}{m_g/d} \right\rfloor B_{g'} - \left\lfloor \frac{kg''(\psi)^{\bullet}}{m_g/d} \right\rfloor (C + B_{g''}) - \sum_{f \neq g, g', g''} \left\lfloor \frac{kf(\psi)^{\bullet}}{m_f} \right\rfloor B_f\,,$$

or linearly equivalent to

$$kL_{\psi} - \left(\left\lfloor \frac{kg''(\psi)^{\bullet}}{m_g/d} \right\rfloor + kq_{\psi}\frac{r_1}{d} + kq'_{\psi} \right) C - \left\lfloor \frac{kg(\psi)^{\bullet}}{m_g} \right\rfloor R - \sum_{f \neq g} \left\lfloor \frac{kf(\psi)^{\bullet}}{m_f} \right\rfloor B_f\,.$$

Now, from (3.3) and (3.1), we get successively

$$\left\lfloor \frac{kg''(\psi)^{\bullet}}{m_g/d} \right\rfloor = \left\lfloor \frac{krg'(\psi)^{\bullet}}{m_g} \right\rfloor - kq'_{\psi} = \left\lfloor \frac{krg(\psi)^{\bullet}}{m_g} \right\rfloor - kq_{\psi}\frac{r}{d} - kq'_{\psi}\,. \qquad \square$$

LEMMA 3.3. *Let $S \to X$ be a normal standard cyclic covering of surfaces defined by the line bundle $\mathcal{L}_{\psi}$ with X smooth. If S has only rational singularities and $\widetilde{S} \to S$ denotes a desingularization of S, then*

$$q(\widetilde{S}) = q(X) + \sum_{j=1}^{n-1} h^1(X, \omega_X \otimes \mathcal{L}_{\psi^j})\,.$$

Proof. Since the singularities are rational, if $\widetilde{S} \xrightarrow{\varepsilon} S$ is a resolution of the singular points of S, then $R^i\varepsilon_*\mathcal{O}_{\widetilde{S}} = 0$, for all $i \geq 1$. From the Leray spectral sequence it follows that $h^i(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = h^i(S, \mathcal{O}_S)$ for all i. Since by Serre duality $q(\widetilde{S}) = h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}})$, we have

$$q(\widetilde{S}) = h^1(S, \mathcal{O}_S) = h^1(X, \pi_*\mathcal{O}_S) = \sum_{j=0}^{n-1} h^1(X, \mathcal{L}_{\psi^j}^{-1})\,.$$

Using the Serre duality, the required equality follows. $\square$

One more notation is in order. Let P be a singular point of B and let $\mu\colon X \to \mathbf{P}^2$ be a log resolution of B at P with $E_{P,1}, E_{P,2}, \dots, E_{P,r}$ the irreducible components of the fibre $\mu^{-1}(P) \subset X$. This finite array of irreducible curves will be denoted by $\boldsymbol{E}_P$. If $\boldsymbol{c}_P$ is a finite array of rational numbers $c_{P,\alpha}$, then

$$\boldsymbol{c}_P \cdot \boldsymbol{E}_P = \sum_{\alpha=1}^{r} c_{P,\alpha} E_{P,\alpha} . \tag{3.5}$$

End of proof of Theorem 3.1. We have seen that if $\mu\colon X \to \mathbf{P}^2$ is a log resolution for B, it is sufficient to compute the irregularity of a desingularization of S_2 which is the pull-back to X of S_1, the standard cyclic covering of the plane defined by $nL'_\psi \sim B + (\lceil b/n \rceil\, n - b)H_\infty$. If the constants $c_{P,\alpha}$ are the multiplicities of the strict transforms of the exceptional divisors that appear in the pull-back of B, i.e. $\mu^* B = \widetilde{B} + \sum_P \boldsymbol{c}_P \cdot \boldsymbol{E}_P$, then the standard cyclic covering S_2 is defined by

$$nL''_\psi \sim \widetilde{B} + (\lceil b/n \rceil\, n - b)\widetilde{H}_\infty + \sum_P \boldsymbol{c}_P \cdot \boldsymbol{E}_P .$$

Notice that $L''_\psi \sim \lceil b/n \rceil\, \widetilde{H}$ and $\widetilde{H}_\infty \sim \widetilde{H}$. By Proposition 3.2, the normalization S of S_2 is defined by the line bundle $\mathcal{L}_\psi$ and

$$\begin{aligned} L_{\psi^k} &\sim kL''_\psi - \left\lfloor \frac{k}{n}(\lceil b/n \rceil\, n - b) \right\rfloor \widetilde{H}_\infty - \sum_P \left\lfloor \frac{k}{n} \boldsymbol{c}_P \right\rfloor \cdot \boldsymbol{E}_P \\ &\sim \left\lceil \frac{kb}{n} \right\rceil \widetilde{H} - \sum_P \left\lfloor \frac{k}{n} \boldsymbol{c}_P \right\rfloor \cdot \boldsymbol{E}_P , \end{aligned} \tag{3.6}$$

the last equality resulting from $\lceil b/n \rceil\, k - \lfloor k(\lceil b/n \rceil\, n - b)/n \rfloor = \lceil kb/n \rceil$. Here, $\lfloor k\boldsymbol{c}_P/n \rfloor \cdot \boldsymbol{E}_P$ denotes $\sum_\alpha \lfloor k c_{P,\alpha}/n \rfloor E_{P,\alpha}$. By Lemma 3.3, $q(S) = \sum_{k=1}^{n-1} h^1(X, K_X + L_{\psi^k})$. Now,

$$K_X + L_{\psi^k} \sim \mu^* K_{\mathbf{P}^2} + \left\lceil \frac{kb}{n} \right\rceil \widetilde{H} + K_{X|\mathbf{P}^2} - \sum_P \left\lfloor \frac{k}{n} \boldsymbol{c}_P \right\rfloor \cdot \boldsymbol{E}_P$$

and

$$\sum_P \left\lfloor \frac{k}{n} \boldsymbol{c}_P \right\rfloor \cdot \boldsymbol{E}_P \sim \left\lfloor \mu^* \frac{k}{n} B \right\rfloor,$$

since the curve $B \subset \mathbf{P}^2$ is reduced. By the local vanishing (2.5), it follows that

$$\begin{aligned} H^1(X, K_X + L_{\psi^k}) &= H^1\Big(X, \mu^* \mathcal{O}_{\mathbf{P}^2}\Big(-3 + \left\lceil \frac{kb}{n} \right\rceil\Big) \otimes \mathcal{O}_X\Big(K_{X|\mathbf{P}^2} - \left\lfloor \mu^* \frac{k}{n} B \right\rfloor\Big)\Big) \\ &\simeq H^1\Big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}\Big(-3 + \left\lceil \frac{kb}{n} \right\rceil\Big) \otimes \mathcal{I}_{Z(\frac{k}{n}B)}\Big), \end{aligned}$$

hence

$$q(S)=\sum_{k=1}^{n-1} h^1\Big(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}\Big(-3+\Big\lceil\frac{kb}{n}\Big\rceil\Big)\otimes\mathcal{I}_{Z(\frac{k}{n}B)}\Big).$$

If $k/n \notin J(B,n)$, then either k/n is not a jumping number of B, or it is, but kb/n is not an integer. In the former case, if ξ is the biggest jumping number for B smaller than k/n, then, since $\lceil kb/n\rceil-\xi>0$,

$$h^1\Big(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}\Big(-3+\Big\lceil\frac{kb}{n}\Big\rceil\Big)\otimes\mathcal{I}_{Z(\frac{k}{n}B)}\Big)=h^1\Big(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}\Big(-3+\Big\lceil\frac{kb}{n}\Big\rceil\Big)\otimes\mathcal{I}_{Z(\xi B)}\Big)=0$$

by the Kawamata-Viehweg-Nadel vanishing theorem. In the latter case, we apply the same argument, now using $\lceil kb/n\rceil - kb/n>0$. The result follows. □

4. Applications

We shall now apply Theorem 3.1 to illustrate how to compute in a uniform way, the irregularity for some examples of cyclic multiple planes. Of course, we shall need to control the multiplier ideals and the jumping numbers attached to the branch curves. In this section we shall deal with curves having singularities only of type A_m, $m\geq 1$. In the appendix, more involved singularities will be considered.

We recall that a singularity of type A_m is defined locally by $x^2+y^{m+1}=0$. The multiplier ideals and their jumping numbers are easy to compute; see for example [4] and [5], or [11]. A different argument for these computations using the theory of clusters will be given in Example A.13.

LEMMA 4.1. *Let B be a curve on a smooth surface and let P be a singular point of B of type A_m. The jumping numbers <1 of B at P are*

$$\xi_a=\frac{1}{2}+\frac{a}{m+1}$$

with $a=1,\dots,\lfloor m/2\rfloor$. If locally around P the curve B is defined by $x^2+y^{m+1}=0$, then, for every a, the multiplier ideal $\mathcal{J}(\xi_a\cdot B)$ is (x,y^a), i.e. the ideal that defines $Z_P^{[a]}$, the 0-dimensional curvilinear subscheme along B supported at P and of length a.

Theorem 3.1 becomes the following:

COROLLARY 4.2. *Let B be a reduced plane curve such that its singularities are either simple nodes or of type A_m with $m \geq 2$ given. Let H_∞ be a line transverse to B and let S be a desingularization of the n-cyclic multiple plane associated to B and H_∞.*

i) *If $m = 2r-1$, then*

$$q(S) = \sum_{\substack{a=1 \\ \frac{a+r}{2r} \in \frac{1}{\gcd(b,n)}\mathbf{Z}}}^{r-1} h^1\Big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}^{[a]}}\Big(-3 + \frac{ab}{2r} + \frac{b}{2}\Big)\Big).$$

ii) *If $m = 2r$, then S may be irregular only if n and b are even, and in this case*

$$q(S) = \sum_{\substack{a=1 \\ \frac{a}{2r+1} \in \frac{1}{\gcd(b,n)}\mathbf{Z}}}^{r} h^1\Big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}^{[a]}}\Big(-3 + \frac{ab}{2r+1} + \frac{b}{2}\Big)\Big).$$

In both formulae, $\mathcal{Z}^{[a]} = \bigcup_P Z_P^{[a]}$.

ZARISKI'S EXAMPLE

The curve B is irreducible, of degree 6 and has six cusps as singularities. If n is divisible by 6, in the formula for the irregularity of the n-cyclic multiple plane from Corollary 4.2ii) we have $a = 1$ since $m = 1$. Hence $q(S) = h^1(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}(2))$, where $\mathcal{Z}$ is the support of the cusps. So either the cusps lie on a conic and the irregularity is 1, or they do not, and the irregularity is 0.

ARTAL-BARTOLO'S FIRST EXAMPLE IN [1]

Let $C \subset \mathbf{P}^2$ be a smooth elliptic curve and let P_1, P_2, P_3 be three inflexion points of C, with L_i the tangent lines at P_i to C. Taking $B = C + L_1 + L_2 + L_3$ we construct the multiple cyclic plane with three sheets S_0 associated to B. The curve B has three points of type A_5 at the P_i's, hence $n = 3$, $b = 6$ and $r = 3$ in Corollary 4.2i). We have $a = 1$ and

$$q(S) = h^1\big(\mathbf{P}^2, \mathcal{I}_{\{P_1, P_2, P_3\}}(1)\big).$$

So the irregularity is 1 if the three inflexion points are chosen on a line. If the points are not on a line, then the irregularity is 0. These two configurations give an example of a Zariski pair.

ARTAL-BARTOLO'S SECOND EXAMPLE IN [1]

Let P be a fixed point and $K = \{P_1, \dots, P_9\}$ a cluster centered at P, all its points being free. It represents a curvilinear subscheme $Z = Z_K$. In [1], Artal-Bartolo considers sextics with an A_{17} type singularity at P, with $P_2, \dots, P_9$ the infinitely near points of the minimal resolution.

1) If P_3 lies on the line L determined by P_1 and P_2 and if K does not impose independent conditions on cubics, then all sextics are reducible. Let B be the union of two smooth cubics from $|\mathcal{I}_Z(3)|$. If S_0 is the 3-cyclic multiple plane associated to B, then by Corollary 4.2 i),

$$q(S) = h^1\big(\mathbf{P}^2, \mathcal{I}_{Z^{[3]}}(1)\big) = 1\,.$$

Similarly, if S_0 is the 6-cyclic multiple plane, then

$$q(S) = h^1\big(\mathbf{P}^2, \mathcal{I}_{Z^{[3]}}(1)\big) + h^1\big(\mathbf{P}^2, \mathcal{I}_{Z^{[6]}}(2)\big) = 2\,,$$

since there is no irreducible conic through $Z^{[6]}$ — i.e. through the points $P_1, \dots, P_6$ — but the double line $2L$: if $K' = \{P_1^2, P_2^2, P_3^2\}$, then $Z^{[6]} \subset Z_{K'}$.

More generally, if S_0 is the n-cyclic multiple plane associated to B and a transverse line H_∞, then by the same argument it follows that $q(S) = 2$ when $n \equiv 0 \mod 6$, $q(S) = 1$ when $n \equiv 3 \mod 6$, and $q(S) = 0$ otherwise.

2) If $P_3 \notin L$ and $P_6 \in \Gamma$, the conic through $P_1, \dots, P_5$, then there exists an irreducible sextic with an A_{17} type singularity at P, such that the intersection with Γ is supported only at P. If S_0 is the n-cyclic multiple plane associated to B and to a transverse line to it, then

$$q(S) = h^1\big(\mathbf{P}^2, \mathcal{I}_{Z^{[6]}}(2)\big) = 1$$

when n is divisible by 6, and $q(S) = 0$ otherwise.

3) If $P_3 \notin L$ and $P_6 \notin \Gamma$, then for every reduced sextic B with an A_{17} type singularity at P, the n-cyclic multiple plane associated to B and to a transverse line to it is regular.

REMARK. In [1] it is shown that in the third case above, two configurations may appear: either $P_1, \dots, P_9$ do not impose independent conditions on cubics and B is the union of two smooth cubics, or the points do impose independent conditions on cubics and B is irreducible. Using these and the two configurations in 1) and 2), two more Zariski couples are thus produced there.

OKA'S EXAMPLE IN [19] WHEN $p = 2$

In [19], if p and q are relatively prime integers, Oka constructs the curve $C_{p,q}$ of degree pq enjoying the following property: $C_{p,q}$ has pq cusp singularities each of which is locally defined by the equation $x^p + y^q = 0$. For the construction, let C_p and C_q be smooth curves of degree p and q that intersect transversely. If $f = 0$ and $g = 0$ are homogeneous equations for C_q and C_p, then C_{pq} is defined globally by $f^p + g^q = 0$.

PROPOSITION 4.3. *The normalization of the pq-multiple plane associated to the curve $C_{p,q}$ is irregular, the irregularity being equal to $(p-1)(q-1)/2$.*

REMARK 4.4. We shall establish the result in the appendix and discuss here the particular case $p = 2$. All the ideas are already present in this situation. In the general computation the argument that uses the trace-residual exact sequence will need the description of the multiplier ideals developed in the appendix and based on the theory of clusters.

Proof when $p = 2$. The integer q must be odd, so let $q = 2r + 1$. To simplify the notation, let $C = C_{2r+1}$ and Γ be the conic transverse to C. The curve $C_{2,2r+1}$ is a curve of degree $4r + 2$ with $4r + 2$ singular points of type A_{2r}. Let S_0 be the $(4r + 2)$-cyclic multiple plane associated to $C_{2,2r+1}$ and let S be the normal cyclic covering constructed in Section 3. We apply Corollary 4.2 ii) to obtain $q(S) = \sum_{\alpha=1}^{r} h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}^{[\alpha]}}(2r + 2\alpha - 2)\big)$, where $\mathcal{Z}^{[\alpha]} = \bigcup_P Z_P^{[\alpha]}$ and $Z_P^{[\alpha]}$ is the curvilinear subscheme associated to the cluster $\{P_1 = P, P_2, \ldots, P_\alpha\}$. We shall apply the trace-residual exact sequence with respect to Γ (see [10]) to show that all the terms of the sum equal 1 and to get $q(S) = r$.

DEFINITIONS. Let X be a projective variety, D be a Cartier divisor on X and ξ be a closed subscheme of X. The schematic intersection $\mathrm{Tr}_D\,\xi = D \cap \xi$ defined by the ideal sheaf $(\mathcal{I}_D + \mathcal{I}_\xi)/\mathcal{I}_D$ is called the *trace* of ξ on D. The closed subscheme $\mathrm{Res}_D\,\xi \subset X$ defined by the conductor ideal $(\mathcal{I}_\xi : \mathcal{I}_D)$ is called the *residual* of ξ with respect to D. The canonical exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}\,\xi}(-D) \longrightarrow \mathcal{I}_\xi \longrightarrow \mathcal{I}_{\mathrm{Tr}\,\xi} \longrightarrow 0$$

is called the *trace-residual exact sequence* of ξ with respect to D.

In our situation, the trace-residual exact sequence with respect to Γ becomes

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}^{[\alpha-1]}}(2r+2\alpha-4) \longrightarrow \mathcal{I}_{\mathcal{Z}^{[\alpha]}}(2r+2\alpha-2) \longrightarrow \mathcal{O}_{\mathbf{P}^1}(4\alpha-6) \longrightarrow 0\,.$$

Since $C \in |\mathcal{I}_{\mathcal{Z}^{[r+1]}}(2r+1)|$, the map $H^0\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}^{[\alpha]}}(2r+2\alpha-2)\big) \longrightarrow H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4\alpha-6)\big)$ from the long exact sequence in cohomology is surjective for every $1 \le \alpha \le r$. Hence

$$h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}^{[r]}}(4r-2)\big) = \cdots = h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}^{[1]}}(2r)\big) = h^1\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2)\big) = 1\,,$$

establishing the proposition in the particular case $p = 2$. □

REMARK. The irregularity of the n-cyclic multiple plane associated to B and to a line H_∞ transverse to B, n being an arbitrary positive integer, may be computed by the same argument. Of course, if $2r+1$ is a prime number, then it might be shown that $q(S) = 0$ unless $4r+2$ divides n — one should use Theorem 3.1 and the result form [22] cited in the introduction. But if $2r+1$ is not a prime number, then irregular cyclic multiple planes exist for other values of n. For example, if $2r+1 = 15$ and $n = 40$, then

$$q(S) = h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}^{[3]}}(18)\big) + h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}^{[6]}}(24)\big) = 2\,.$$

A SPECIALIZATION OF OKA'S EXAMPLE WHEN $p = 2$

Keeping the notation from the previous paragraph, the conic Γ is now the union of two distinct lines that intersect at O and C is a smooth curve of degree $2r+1$ passing through O and intersecting transversely the lines of Γ at this point. The curve B has $4r$ points of type A_{2r} and one singular point at O of type A_{4r+1}. It can be shown that the irregularity of the $(4r+2)$-cyclic multiple plane associated to B is again r. We develop the computation for $r = 2$. In this case, B is a curve of degree 10 with 8 points of type A_4 and one point of type A_9. By Theorem 3.1 and using the notation from Corollary 4.2, the irregularity is given by

$$h^1\big(\mathbf{P}^2, \mathcal{I}_{\xi^{[1]} \cup Z_O^{[2]}}(4)\big) + h^1\big(\mathbf{P}^2, \mathcal{I}_{\xi^{[2]} \cup Z_O^{[4]}}(6)\big)\,,$$

where $\xi^{[1]}$ is the support of the points of type A_4 and $\xi^{[2]} = \bigcup_{P \text{ of type } A_4} Z_P^{[2]}$ is the support plus the tangent directions. Now, 10 points on a conic do not impose independent conditions on quartics, hence the first term is 1. The second term is seen to be equal to the first after applying the trace-residual exact sequence with respect to the two lines of Γ. So the irregularity is 2.

The computations for $r = 1$ lead to a branching curve of degree 6 with four cusps and an A_5 singularity at O. The irregularity of a 6-cyclic multiple plane is 1, given by $h^1\big(\mathbf{P}^2, \mathcal{I}_{\xi^{[1]} \cup Z_O^{[2]}}(2)\big)$. If in addition, the two lines of the degenerate conic Γ are brought together such that the cusps collapse two by two, the branching curve has three A_5 singularities. For a 6-multiple plane, $q = 2$, with the contributions of the superabundance of the singularities with respect to the lines and the conics both equal to 1. The branching curve is reducible; it is Artal-Bartolo's first example.

LINE ARRANGEMENTS FOLLOWING [7]

In this example we consider as branch curve a line arrangement $B = \bigcup_{i=1}^{b} L_i \subset \mathbf{P}^2$ that has only nodes and ordinary triple points as singularities. We revisit, from the point of view developed here, results obtained in [7]. See also [2] where line arrangements are examined using the techniques from [1].

Using Example 2.8 or Corollary A.2, we have that for an ordinary triple point $2/3$ is the only jumping number < 1. The multiplier ideal is $\mathcal{I}_P$. By Theorem 3.1, if H_∞ is a line transverse to $B = \bigcup_{i=1}^{b} L_i$, then the normal n-cyclic covering S corresponding to the n-cyclic multiple plane associated to B and H_∞ is irregular if and only if 3 divides both b and n, and $\left|\mathcal{I}_{\mathcal{Z}}(-3 + \frac{2b}{3})\right|$ is superabundant, in which case

$$q(S) = h^1\left(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}\left(-3 + \frac{2b}{3}\right)\right).$$

In case S is irregular, it can be shown that the irregularity is bounded by a constant depending on the arrangement B.

PROPOSITION 4.5. *Let $B = \bigcup_{i=1}^{b} L_i$, H_∞ and S be as above with b and n divisible by 3. If t_i is the number of triple points lying on the line L_i for each i, then*

$$q(S) \leq \min_{i=1,\ldots,b} t_i .$$

For the proof (see [7] for a different argument), we need a preliminary lemma.

LEMMA 4.6. *If 3 divides both b and n and if one line of the arrangement contains no triple point, then $q(S) = 0$.*

Proof. Let B' be the arrangement of the $b-1$ lines of B except the one with no triple point. If S' is the normal n-cyclic covering corresponding to the n-cyclic multiple plane associated to B' and H_∞, then $q(S') = 0$ since 3 does not divide $\deg B'$. Taking $k = 2n/3$ in Theorem 3.1 we obtain

$$0 = h^1\Big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}\Big(-3 + \Big\lceil\frac{2(b-1)}{3}\Big\rceil\Big)\Big) = h^1\Big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}\Big(-3 + \frac{2b}{3}\Big)\Big) = q(S)$$

where $\mathcal{Z}$ is the support of the triple points. $\square$

Proof of Proposition 4.5. Let us suppose that L_1 is the line containing the minimum number of triple points. If $B' = L_1' \cup \bigcup_{i\neq 1} L_i$ is a line arrangement with no triple point on L_1' and if $\mathcal{Z}'$ denotes the support of the triple points of B', then by the previous lemma, $h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}'}(-3+2b/3)\big) = 0$. Since $h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}(-3+2b/3)\big)$ measures how much $\mathcal{Z}$ fails to impose independent conditions of the curves of degree $2b/3-3$,

$$h^1\Big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}\Big(-3+\frac{2b}{3}\Big)\Big) \le h^1\Big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}'}\Big(-3+\frac{2b}{3}\Big)\Big) + \operatorname{card}(\mathcal{Z}-\mathcal{Z}') = t_1,$$

hence the result. $\square$

EXAMPLE. Let B be the line arrangement of 9 lines with 9 triple points represented below (Figure 2). In a convenient affine coordinate system (x,y), the triple points that lie in the affine plane are the following:
$(0,0)$, $(\pm 2,-2)$, $(-2,0)$, $(0,s)$, $(2,s)$ and $\dfrac{2s}{(s+4)}(-1,1)$, with $s \neq -2, 0$ and 2.

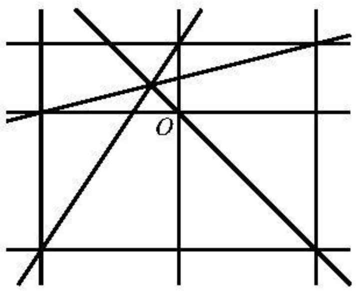

FIGURE 2

It is easy to see that there are two cubics — each the union of three lines — through the 9 triple points, i.e. the system of cubics through the points is superabundant. It follows that the irregularity of the n-cyclic multiple plane associated to B and to a line H_∞ transverse to B, is 1 if and only if 3 divides n.

If $s = 2$, then the arrangement specializes to an arrangement with 10 triple points, 4 of them lying on the line $x+y=0$. But these points lie on a cubic, the union of three of the lines of B, and again $h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}(3)\big) = 1$, hence the irregularity is 1 in this case too.

REMARK 4.7. The irregularity depends on the position of the line H_∞ with respect to B. To see this, let B be the line arrangement below of 5 lines with 2 triple points from Figure 3.

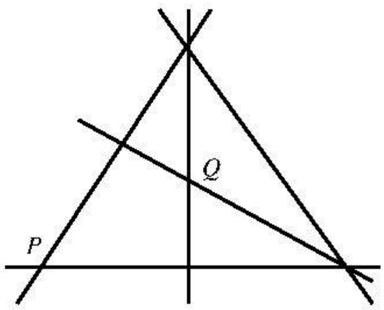

FIGURE 3

If H_∞ is transverse to B, then the irregularity of the 6-cyclic multiple plane is 0. But if H_∞ is the line through the double points P and Q then the irregularity jumps to 1.

A. CLUSTERS AND MULTIPLIER IDEALS

Among the examples treated in Section 4 there is Oka's example. The irregularity of the surface involved is computed in Proposition 4.3. The proposition was proved only in the particular case when the singularities of the branch curve are of type A_m. The general proof may be supplied along the lines developed in the particular case on condition that the multiplier ideals involved in the formula for the irregularity have a description fit for use in the trace-residual exact sequence.

Throughout this appendix we work under the following hypothesis: B is a curve on a smooth surface such that each of its singular points is locally characterized by an equation of type $x^m + y^n = 0$. We shall give a new description of the multiplier ideals attached to B. They are determined only by the study of the coefficients of the last exceptional curves in a log resolution of B.

PROPOSITION A.1. *Let $\mu\colon Y \to X$ be a log resolution of B in X and denote by E_P the last exceptional curve in the resolution above each singular point P. If c_P is the coefficient of E_P in $-K_{Y|X} + \lfloor \mu^* \xi B \rfloor$, then the multiplier ideal $\mathcal{J}(\xi \cdot B)$ is given by*

$$\mathcal{J}(\xi \cdot B) = \mu_* \bigotimes_P \mathcal{O}_Y(-c_P E_P)\,.$$

Moreover, if P is locally given by $x^m + y^n = 0$ with $d = \gcd(m,n)$ and $m = dp$, $n = dq$, then

$$\mu_* \mathcal{O}_Y(-c_P E_P) = \mu_* \mathcal{O}_Y(-\bar{c}_P E_P)\,, \quad \text{with} \quad \bar{c}_P = \min_{\substack{ap+bq \geq c_P \\ a,b \geq 0}} (ap + bq)\,,$$

and $\mu_* \mathcal{O}_Y(-c E_P) \subsetneq \mu_* \mathcal{O}_Y(-\bar{c}_P E_P)$ for any $c > \bar{c}$.

We refer the reader to [11] for a different description of these multiplier ideals.

COROLLARY A.2 (see [4, 5, 11]). *Let P be a singular point of a curve B on a smooth surface S. If P is locally given by $x^m + y^n = 0$, then the jumping numbers of B at P are*

$$\frac{a}{m} + \frac{b}{n}$$

with a and b positive integers.

Above a singular point P, through the log resolution μ, lies an exceptional configuration, a $\mathbf{Z}$-linear combination of strict transforms of exceptional divisors. The proof of Proposition A.1 will mainly deal with this configuration. To prepare the way for the proof we need to formalize the setup and recall some results from the theory of clusters.

A.1 CLUSTERS AND ENRIQUES DIAGRAMS

Let X be a surface and $P \in X$ a smooth point. A point Q is called *infinitely near* to P if $Q \in X'$ with $\mu\colon X' \to X$ a composition of blowing ups and Q lying on the exceptional configuration that maps to P. The points infinitely near to P are partially ordered. The point Q precedes the point R if and only if R is infinitely near to Q.

DEFINITION. A *cluster* in X centered at a smooth point P is a finite set of weighted infinitely near points to P, $K = \{P_1^{w_1}, \dots, P_r^{w_r}\}$, with $P_1 = P$ and such that the ordering of the points is compatible with the partial order of the infinitely near points — if $\alpha < \beta$ then either P_β is infinitely near to P_α or there is $\gamma < \alpha$ such that P_α and P_β are infinitely near to P_γ. The point P_1 is called the *proper* point of the cluster.

In the sequel, if K is a cluster, then all points preceding a point that belongs to K will belong to K, possibly with weight 0.

Let $Y = Y_{r+1} \to Y_r \to \cdots \to Y_1 = X$ be the decomposition of $\mu\colon Y \to X$ into successive blowing ups with $Y_{\alpha+1} = \mathrm{Bl}_{P_\alpha} Y_\alpha$. Each point P_α corresponds to an exceptional divisor $E_\alpha \subset Y_{\alpha+1}$. All its strict transforms will also be denoted by E_α and the total transform of each E_α by W_α. When needed, the strict transform of E_α on Y_β will be denoted by $E_\alpha^{(\beta)}$, and similarly for the total transform. For example $W_\alpha^{(\alpha+1)} = E_\alpha^{(\alpha+1)}$.

The strict transforms E_α and the total transforms W_α form two different bases of the $\mathbf{Z}$-module $\bigoplus_\alpha \mathbf{Z}E_\alpha \subset \operatorname{Pic} Y$. The combinatorics of the configuration of the strict transforms on Y is codified in the notion of *proximity* for the points of the cluster: a point P_β is said to be *proximate* to P_α, $P_\beta \prec P_\alpha$, if P_β lies on $E_\alpha^{(\beta)} \subset Y_\beta$, the strict transform on Y_β of the exceptional divisor corresponding to the blow-up at P_α. Besides, a point that is infinitely near, i.e. that is not proper, is always proximate to at most two other points of the cluster. It is said to be *free* if it is proximate to exactly one point and *satellite* if it is proximate to exactly two points of the cluster.

Let $\Pi = \|p_{\alpha\beta}\|$ be the decomposition matrix of the strict transforms in terms of the total transforms on Y. Since

$$E_\alpha = W_\alpha - \sum_{P_\beta \prec P_\alpha} W_\beta,$$

$p_{\alpha\alpha} = 1$ for any α and $p_{\alpha\beta}$ equals -1 if P_β is proximate to P_α and 0 otherwise. Notice that along the α column of Π the non-zero elements not on the diagonal correspond to the points to which P_α is a satellite. The matrix $-\Pi \cdot {}^t\Pi$ is the intersection matrix of the curves E_α on the surface Y. For any α,

$$E_\alpha^2 = -(1 + p_\alpha),$$

where p_α is the number of points P_β proximate to P_α. Since the intersection matrix of the curves W_α is minus the identity, there exist effective divisors B_α that form the dual basis for the divisors $-E_\alpha$'s with respect to the intersection form. In the sequel this basis will be referred to as the *branch basis*. Clearly, the decomposition matrix of the basis of strict transforms in terms of the branch basis is $\Pi \cdot {}^t\Pi$.

The points of a cluster K, their weights and proximity relations were encoded by Enriques in a convenient tree diagram now called the *Enriques diagram* of the cluster (see [3, 6, 8]). If the weights are omitted, the tree reflects the combinatorics of the configuration of the strict transforms $E_\alpha \subset Y$.

DEFINITION. An *Enriques tree* is a couple (T, ε), where $T = T(\mathfrak{V}, \mathfrak{E})$ is an oriented tree (a graph without loops) with a single *root*, with $\mathfrak{V}$ the set of vertices and $\mathfrak{E}$ the set of edges, and where ε is a map

$$\varepsilon \colon \mathfrak{E} \longrightarrow \{\text{'slant'}, \text{'horizontal'}, \text{'vertical'}\}$$

fixing the graphical representation of the edges. An *Enriques diagram* is a weighted Enriques tree.

EXAMPLE A.3. Let $p < q$ be relatively prime positive integers. $T_{p,q}$ will denote the Enriques tree associated to the Euclidean algorithm. It is a unibranch tree. Let $r_0 = a_1 r_1 + r_2, \ldots, r_{m-2} = a_{m-1} r_{m-1} + r_m$ and $r_{m-1} = a_m r_m$, with $r_0 = q$ and $r_1 = p$. The oriented tree has $\mathfrak{V} = \{P_\alpha \mid 1 \le \alpha \le a_1 + \cdots + a_m\}$ and $\mathfrak{A} = \{[P_\alpha P_{\alpha+1}] \mid 1 \le \alpha \le a_1 + \cdots + a_m - 1\}$. The map ε is locally constant on the a_j edges $[P_\alpha P_{\alpha+1}]$ with $a_1 + \cdots + a_{j-1} + 1 \le \alpha \le a_1 + \cdots + a_j$. The first constant value of ε — on the first a_1 edges — is 'slant'. The other constant values are alternatively either 'horizontal' or 'vertical', starting with 'horizontal'. The Enriques trees $T_{1,3}$, $T_{2,3}$ and $T_{5,7}$ are represented in Figure 4. The tree $T_{5,7}$ together with the weights $w_1 = 5$, $w_2 = w_3 = 2$

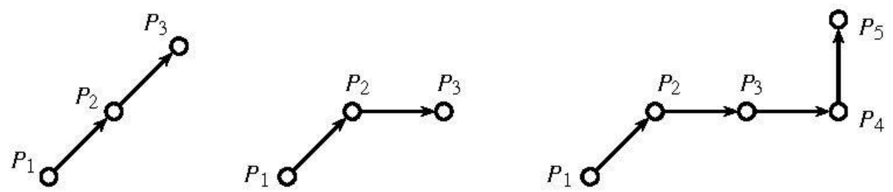

FIGURE 4

and $w_4 = w_5 = 1$ becomes the Enriques diagram that reflects the Euclidean algorithm for $p = 5$ and $q = 7$: if P_α is the initial vertex of one of the a_j edges on which ε has constant value, then $w_\alpha = r_j$. The configuration of exceptional curves is the configuration obtained when desingularizing the curve $x^5 + y^7 = 0$.

The fact that clusters and Enriques diagrams carry the same information is asserted by the following lemma. One more piece of terminology first. Let T be an Enriques tree. A horizontal (respectively vertical) *L-shaped branch* of T is an ordered chain of edges such that the final vertex of each is the starting vertex of the next, and such that all edges, but the first, are horizontal (respectively vertical) through ε. An edge is an L-shaped branch, regardless its value through g. It is a horizontal L-shaped branch if its value through ε is either slant or vertical and it is a vertical L-shaped branch if its value through ε is horizontal.

An L-shaped branch is proper if it contains at least two edges. A *maximal L-shaped branch* is an L-shaped branch that can not be continued to a longer one.

LEMMA A.4 (see [8], Proposition 1.2). *There exists a unique map from the set of clusters in X centered at a smooth point P to the set of Enriques diagrams such that:*

- *For every cluster $K = \{P_1^{w_1}, \ldots, P_r^{w_r}\}$ the set of vertices of the image tree is $\mathfrak{V} = \{P_1, \ldots, P_r\}$ with the weights given by the integers $w_1, w_2, \ldots, w_r$. The root of the tree is the proper point.*
- *At every point there ends at most one edge.*
- *A point P_α is satellite if and only if there is either a horizontal or a vertical edge that ends at the vertex P_α.*
- *If there is an edge that begins at the vertex P_α and ends at the vertex P_β then $P_\beta \in E_\alpha^{(\beta)}$, and the converse is true if P_β is free.*
- *The point P_β is proximate to P_α if and only if there is an L-shaped branch that starts at P_α and ends at P_β.*
- *The strict transforms E_α and E_β intersect on Y if and only if the Enriques diagram contains a maximal L-shaped branch that has P_α and P_β as its extremities.*
- *An edge that begins at a vertex of a free point and ends at a vertex of a satellite point is horizontal.*

EXAMPLE A.5. The Enriques tree $T_{5,7}$ seen in the previous example has two maximal horizontal L-shaped branches. These branches are shown in Figure 5 together with the configuration of the strict transforms of the exceptional curves.

A.2 UNLOADED CLUSTERS

Let $K = \{P_1^{w_1}, \ldots, P_r^{w_r}\}$ be a cluster centered at P. It defines a divisor $D_K = \sum w_\alpha W_\alpha$ on Y, an ideal sheaf $\mu_* \mathcal{O}_Y(-D_K)$ on X and hence a

FIGURE 5

subscheme Z_K of X. The decomposition matrix Π is also called the *proximity matrix* of the cluster. Using it,

$$D_K = \sum_\alpha w_\alpha W_\alpha = \sum_\alpha c_\alpha E_\alpha = \sum_\alpha b_\alpha B_\alpha ,$$

with $\boldsymbol{w} = \boldsymbol{c} \cdot \Pi$ and $\boldsymbol{b} = \boldsymbol{c} \cdot \Pi \cdot {}^t\Pi$, where $\boldsymbol{w} = (w_1, \dots, w_r)$, $\boldsymbol{c} = (c_1, \dots, c_r)$ and similarly $\boldsymbol{b} = (b_1, \dots, b_r)$. The lemma below clarifies the comparison between the divisor D_K and the ideal sheaf $\mu_* \mathcal{O}_Y(-D_K)$ or, equivalently, the subscheme Z_K.

LEMMA A.6. *Let $D_K = \sum_\alpha b_\alpha B_\alpha$. If $b_\beta < 0$ for a certain β, then*

$$\mu_* \mathcal{O}_Y(-D_K) = \mu_* \mathcal{O}_Y(-D_K - E_\beta) .$$

Proof. We take μ_* on the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-D_K - E_\beta) \longrightarrow \mathcal{O}_Y(-D_K) \longrightarrow \mathcal{O}_{E_\beta}(-D_K \mid_{E_\beta}) \longrightarrow 0 .$$

Since

$$\deg(-D_K \mid_{E_\beta}) = -\Big(\sum b_\alpha B_\alpha\Big) \cdot E_\beta = b_\beta < 0$$

we have $\mu_* \mathcal{O}_{E_\beta}(-D_K \mid_{E_\beta}) = 0$. $\square$

A cluster K is said to satisfy the *proximity relations* if for every P_α in K,

$$\overline{w}_\alpha = \sum_{P_\beta \prec P_\alpha} w_\beta \le w_\alpha .$$

COROLLARY A.7 (see [3], Theorem 4.2). *Let $K = \{P_1^{w_1}, \dots, P_r^{w_r}\}$ be a cluster that contains a point P_α at which the proximity relation is not satisfied. If $K' = \{P_1^{w_1'}, \dots, P_r^{w_r'}\}$ is the cluster defined by $w_\alpha' = w_\alpha + 1$, $w_\beta' = w_\beta - 1$ for every β with P_β proximate to P_α, and $w_\gamma' = w_\gamma$ otherwise, then K and K' define the same subscheme in X, i.e. $\mu_* \mathcal{O}_Y(-D_K) = \mu_* \mathcal{O}_Y(-D_{K'})$.*

Proof. Let $D_K = \sum_\alpha w_\alpha W_\alpha = \sum c_\alpha E_\alpha = \sum_\alpha b_\alpha B_\alpha$ and $D_{K'} = \sum_\alpha b'_\alpha B_\alpha$. The coefficients b_α are given by $\boldsymbol{b} = \boldsymbol{c} \cdot \Pi \cdot {}^t\Pi = \boldsymbol{w} \cdot {}^t\Pi = \boldsymbol{w} - \overline{\boldsymbol{w}}$. Then

$$\boldsymbol{b}' = \boldsymbol{w}' - \overline{\boldsymbol{w}}' = \boldsymbol{w} - \overline{\boldsymbol{w}} + (\Pi \cdot {}^t\Pi)_\alpha = \boldsymbol{b} + (\Pi \cdot {}^t\Pi)_\alpha$$

and

$$\begin{aligned} \boldsymbol{c}' = \boldsymbol{b}' \cdot (\Pi \cdot {}^t\Pi)^{-1} &= \boldsymbol{b} \cdot (\Pi \cdot {}^t\Pi)^{-1} + (\Pi \cdot {}^t\Pi)_\alpha \cdot (\Pi \cdot {}^t\Pi)^{-1} \\ &= \boldsymbol{c} + (0, \dots, 1, \dots, 0), \end{aligned}$$

hence $D_{K'} = D_K + E_\alpha$. But $b_\alpha = w_\alpha - \overline{w}_\alpha < 0$ and the result follows from the previous lemma. □

The cluster K' is said to be obtained from K by the *unloading procedure*. Starting from K, iterated applications of this procedure lead to a cluster $\overline{K}$ that satisfies the proximity relations and defines the same subscheme in X. The cluster $\overline{K}$ is called the *unloaded cluster* associated to K. Notice that a cluster is unloaded if and only if the coefficients of its divisor in the branch basis are non-negative.

EXAMPLE. Let $\{P_1^5, P_2^2, P_3^2, P_4^1, P_5^1\}$ and $\{P_1^4, P_2^2, P_3^0, P_4^2, P_5^1\}$ be two clusters with the proximity encoded by the Enriques tree $T_{5,7}$. The former is unloaded. The latter does not satisfy the proximity relation at P_3. The unloaded associated cluster $\overline{K}_2$ is $\{P_1^4, P_2^2, P_3^1, P_4^1, P_5^0\}$ and $D_{\overline{K}_2} = B_2 + B_4$.

A.3 THE PROOF OF PROPOSITION A.1

We consider $\mathfrak{K}_{p,q}$, the set of unloaded clusters whose lattice tree is $T_{p,q}$ with $p < q$ relatively prime positive integers. Given a positive integer c, we want to characterize the unloaded cluster in $\mathfrak{K}_{p,q}$ whose associated ideal is $\mu_* \mathcal{O}_Y(-cE_r)$, with E_r the last exceptional divisor.

All along this subsection the W_α's, $1 \le \alpha \le r$, denote the total transforms of the exceptional divisors in the process of blowing up the points of the clusters in $\mathfrak{K}_{p,q}$, the E_α's the strict transforms and the B_α's the elements of the branch basis. The integer r is given by $r = a_1 + \cdots + a_m$, where $q = a_1 p + r_2$, $p = a_2 r_2 + r_3$, ..., $r_{m-1} = a_m r_m$. Finally, $\varphi \colon \bigoplus_\alpha \mathbf{Z}E_\alpha \to \mathbf{Z} \simeq \mathbf{Z}E_r$ denotes the projection of $\bigoplus_\alpha \mathbf{Z}E_\alpha$ on its last factor.

We need four lemmas. In the following lemma two finite sequences closely linked to the Euclidean algorithm are introduced.

LEMMA A.8. *If $(f_j)_{-1\le j\le m}$ and $(\delta_j)_{1\le j\le m+1}$ are two finite sequences defined by*

$$f_j = f_{j-2} + a_j\delta_j \quad \text{for any } 1 \le j \le m$$

and

$$\delta_j = \delta_{j-2} + a_{j-1}f_{j-2} \quad \text{for any } 2 \le j \le m+1$$

and such that $f_{-1} = f_0 = 0$ and $\delta_0 = \delta_1 = 1$, then the remainder r_j in the Euclidean algorithm is given by $-f_{j-1}q + \delta_j p$ if j is odd and $\delta_j q - f_{j-1}p$ if j is even.

Proof. Left to the reader. □

REMARK A.9. If m is odd, then $f_m = q$ and $\delta_{m+1} = p$, and if m is even, then $f_m = p$ and $\delta_{m+1} = q$. Indeed, let us suppose that m is odd. Then the equalities follow since for any $1 \le j \le m$ the integers f_j and δ_{j+1} are relatively prime and $0 = r_{m+1} = \delta_{m+1}q - f_m p$.

LEMMA A.10. *If $(f_j)_{-1\le j\le m}$ and $(\delta_j)_{1\le j\le m+1}$ are the finite sequences defined in Lemma A.8, then for any $1 \le j \le m$ and any $1 \le k \le a_j$, the coefficient of the last strict transform in $B_{a_1+\cdots+a_{j-1}+k}$ equals either $(f_{j-2} + k\,\delta_j)p$ if j is odd or $(f_{j-2} + k\,\delta_j)q$ if j is even.*

Proof. The proof proceeds by induction on j and k. It is clear for $j = 1$ and any k. Suppose that j is even, $k < a_j$ and that $\varphi(B_{a_1+\cdots+a_{j-1}+k}) = (f_{j-2} + k\,\delta_j)q$. We recall that B_α is given by the Enriques diagram for which the weight of the point P_α is $w_\alpha = 1$, the weights of all the points that do not precede P_α are 0, and all the others are computed by imposing equalities in the proximity relations. Then

$$B_{a_1+\cdots+a_{j-1}+k+1} = B_{a_1+\cdots+a_{j-1}+k} + W_{a_1+\cdots+a_{j-1}+k+1} + B_{a_1+\cdots+a_{j-1}}$$

and

$$\begin{aligned}\varphi(B_{a_1+\cdots+a_{j-1}+k+1}) &= (f_{j-2} + k\,\delta_j)q + r_j + (f_{j-3} + a_{j-1}\,\delta_{j-1})p\\ &= (f_{j-2} + k\,\delta_j)q + \delta_j q - f_{j-1}p + f_{j-1}p\\ &= (f_{j-2} + (k+1)\delta_j)q\,.\end{aligned}$$

The argument is similar in all the other cases, i.e. when either $k = a_j$ or j odd. □

LEMMA A.11. *If $K \in \mathfrak{K}_{p,q}$, then the coefficient of E_r in D_K is of the form $ap + bq$, with a, b non-negative integers.*

Proof. Since $D_K = \sum_\alpha (w_\alpha - \overline{w}_\alpha) B_\alpha$ the result follows from the previous lemma. □

NOTATION. The cluster $K_{p,q}(ap+bq)$, $a, b \geq 0$, is the unloaded cluster associated to $\{P_1^0, \ldots, P_{r-1}^0, P_r^{ap+bq}\}$, i.e. whose associated ideal is $\mu_* \mathcal{O}_Y(-(ap+bq)E_r)$.

LEMMA A.12. *Let $K \in \mathfrak{K}_{p,q}$ such that $\varphi(D_K) = ap + bq$. Then $K = K_{p,q}(ap+bq)$, the cluster that corresponds to $\mu_* \mathcal{O}_Y\big(-(ap+bq)E_r\big)$, if and only if every ordered chain of maximal L-shaped branches determined by the points $P_{\alpha_1}, \ldots, P_{\alpha_l}$ — each P_{α_k} precedes $P_{\alpha_{j+1}}$ and the jth maximal L-shaped branch starts at P_{α_k} and ends at $P_{\alpha_{k+1}}$ — satisfies*

$$(*) \qquad \sum_{k=1}^{l} (w_{\alpha_k} - \overline{w}_{\alpha_k}) < \sum_{k=1}^{l} p_{\alpha_k} + 2 - l\,.$$

(Recall that the non-negative integer p_α is the number of points P_β that are proximate to P_α.)

Proof. The proof divides into four steps the third being the main one. First, if an unloaded cluster does not satisfy the condition $(*)$, then an inverse of the unloading procedure may be applied to K with the output an unloaded cluster. Indeed, suppose that there exists an ordered chain of maximal L-shaped branches determined by the points $P_{\alpha_1}, \ldots, P_{\alpha_l}$ such that $\sum_{k=1}^{l} (w_{\alpha_k} - \overline{w}_{\alpha_k}) \geq \sum_{k=1}^{l} p_{\alpha_k} + 2 - l$. We may further assume that all its proper subchains satisfy $(*)$. It follows that

$$w_{\alpha_1} - \overline{w}_{\alpha_1} = p_{\alpha_1}\,, \quad w_{\alpha_l} - \overline{w}_{\alpha_l} = p_{\alpha_l}$$

and for any other α_k,

$$w_{\alpha_k} - \overline{w}_{\alpha_k} = p_{\alpha_k} - 1\,.$$

By Lemma A.4, the strict transforms $E_{\alpha_1}, \ldots, E_{\alpha_l}$ intersect two by two. Then,

$$\begin{aligned}\Big(D_K - \sum_{k=1}^{l} E_{\alpha_k}\Big) \cdot E_{\alpha_i} &= \Big(\sum_\alpha b_\alpha B_\alpha - \sum_{k=1}^{l} E_{\alpha_k}\Big) \cdot E_{\alpha_i} \\ &= \begin{cases} -b_{\alpha_1} + (p_{\alpha_1} + 1) - 1\,, & i = 1 \\ -b_{\alpha_i} - 1 + (p_{\alpha_i} + 1) - 1\,, & 1 < i < l \\ -b_{\alpha_l} - 1 + (p_{\alpha_l} + 1)\,, & i = l \end{cases}\end{aligned}$$

hence $(D_K - \sum_{k=1}^{l} E_{\alpha_k}) \cdot E_{\alpha_i} \leq 0$ for any α_i. We conclude that the cluster K' whose divisor is $D_K - \sum_{k=1}^{l} E_{\alpha_k}$ is still unloaded and the coefficient of E_r in $D_{K'}$ is unchanged. Clearly, $\mu_* \mathcal{O}_Y(-D_K) \subset \mu_* \mathcal{O}_Y(-D_{K'})$.

Second, if K is a cluster that satisfies $(*)$, then for any $1 \leq \beta \leq r$ and any ordered chain of maximal L-shaped branches determined by the points $P_{\alpha_1}, \ldots, P_{\alpha_l}$ such that the last one ends in P_β,

$$\sum_{j=1}^{l} b_{\alpha_j} \varphi(B_{\alpha_j}) \leq \varphi(B_\beta). \tag{A.1}$$

Let us suppose that P_β is the final vertex of a maximal horizontal L-shaped branch (the argument being similar if the branch is vertical). There are two cases: the last maximal L-shaped branch is either proper or it is not, i.e. there exists $i \leq r-1$ such that $\beta = a_1 + \cdots + a_i + k$ with in the first case $k = 1$ and in the second $2 \leq k \leq a_{i+1}$ (see Figure 6).

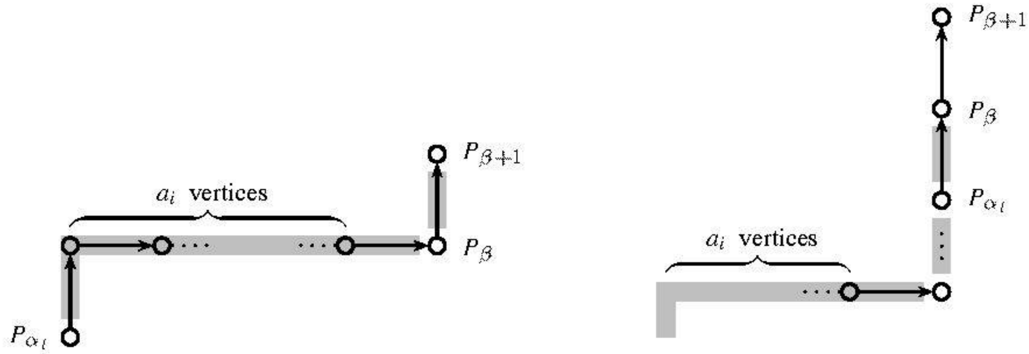

FIGURE 6

In the first case, due to the condition $(*)$ and using the formulae of Lemma A.8,

$$\begin{aligned}
\sum_{j=1}^{l} b_{\alpha_j} \varphi(B_{\alpha_j}) &\leq \sum_{j=1}^{l-1} (p_{\alpha_j} - 1) \varphi(B_{\alpha_j}) + p_{\alpha_l} \varphi(B_{\alpha_l}) \\
&\leq (a_2 f_1 + a_4 f_3 + \cdots + (a_i + 1) f_{i-1}) p \\
&= (-\delta_1 + \delta_1 + a_2 f_1 + a_4 f_3 + \cdots + (a_i + 1) f_{i-1}) p \\
&= (-\delta_1 + \delta_{i+1} + f_{i-1}) p \\
&= -p + \varphi(B_{a_1 + \cdots + a_i + 1}).
\end{aligned}$$

For the second inequality we have supposed that $\beta < r$ and hence $p_{\alpha_l} = p_{a_1+\cdots+a_{i-1}} = a_i + 1$. If $\beta = r$, then $\alpha_l = a_1 + \cdots + a_{m-1}$ and $p_{\alpha_l} = a_m$ and we get

$$\sum_{j=1}^{l} b_{\alpha_j}\varphi(B_{\alpha_j}) \leq (a_2 f_1 + a_4 f_3 + \cdots + a_m f_{m-1})p = (-\delta_1 + \delta_{m+1})p = -p + \varphi(B_r).$$

Similarly, in the second case, i.e. when $\beta = a_1 + \cdots + a_i + k$ with $2 \leq k \leq a_{i+1}$,

$$\begin{aligned}\sum_{j=1}^{l} b_{\alpha_j}\varphi(B_{\alpha_j}) &\leq \sum_{j=1}^{l-1}(p_{\alpha_j} - 1)\varphi(B_{\alpha_j}) + p_{\alpha_l}\varphi(B_{\alpha_l})\\ &\leq (a_2 f_1 + a_4 f_3 + \cdots + a_i f_{i-1} + (f_{i-1} + (k-1)\delta_{i+1}))p\\ &= (-\delta_1 + \delta_{i+1} + (f_{i-1} + (k-1)\delta_{i+1}))p\\ &= -p + \varphi(B_{a_1+\cdots+a_i+k}).\end{aligned}$$

Third, if both K and K' are unloaded and satisfy the condition $(*)$ and are such that $\varphi(D_K) = \varphi(D_{K'})$, then $K = K'$. To justify this, let $D_K = b_1 B_1 + \cdots + b_r B_r$ and $D_{K'} = b'_1 B'_1 + \cdots + b'_r B'_r$. We want to show that for any α, $b_\alpha = b'_\alpha$. Then, if $P_\alpha \rightsquigarrow \mathrm{L}_h$ indicates that the vertex P_α is the initial vertex of a maximal horizontal L-shaped branch,

$$\varphi(D_K) = \sum_{\substack{\alpha<r\\ P_\alpha \rightsquigarrow \mathrm{L}_h}} b_\alpha\varphi(B_\alpha) + \sum_{\substack{\alpha<r\\ P_\alpha \rightsquigarrow \mathrm{L}_v}} b_\alpha\varphi(B_\alpha) + b_r\varphi(B_r) = ap + bq + b_r pq,$$

with a, b non-negative integers. By the previous step applied to $\beta = r$, it follows that $q > a$ and $p > b$. Analogously, $\varphi(D_{K'}) = a'p + b'q + b'_r pq$, with $q > a' \geq 0$ and $p > b' \geq 0$. We get

$$(b'_r - b_r)pq = (a - a')p + (b - b')q$$

with $|a - a'| < p$ and $|b - b'| < q$, and hence $b_r = b'_r$, $a = a'$ and $b = b'$ since the integers p and q are relatively prime. Now, the equality among the other coefficients is established similarly. Keeping the above notation, suppose that $b_\alpha = b'_\alpha$ for any $\alpha > \beta$, with $\beta < r$ the initial edge of a maximal horizontal L-shaped branch. If $b_\beta < b'_\beta$, then

$$\begin{aligned}ap &= \sum_{\substack{\alpha<\beta\\ P_\alpha \rightsquigarrow \mathrm{L}_h}} b_\alpha\varphi(B_\alpha) + b_\beta\varphi(B_\beta) + \sum_{\substack{\beta<\alpha<r\\ P_\alpha \rightsquigarrow \mathrm{L}_h}} b_\alpha\varphi(B_\alpha)\\ &< \varphi(B_\beta) + b_\beta\varphi(B_\beta) + \sum_{\substack{\beta<\alpha<r\\ P_\alpha \rightsquigarrow \mathrm{L}_h}} b'_\alpha\varphi(B_\alpha) \leq a'p,\end{aligned}$$

contradicting the identity $a = a'$ obtained previously.

To finish the proof of the lemma, we notice that the unloaded cluster whose associated ideal is $\mu_* \mathcal{O}_Y\big(-(ap+bq)E_r\big)$ satisfies the condition $(*)$ since the subscheme supported at $P = P_1$ defined by this ideal is the smallest subscheme such that its pull-back on Y contains E_r with multiplicity $ap+bq$. We have seen in the first step that $(*)$ characterizes this minimality condition. □

Now we are ready to identify the unloaded cluster whose associated ideal is $\mu_* \mathcal{O}_Y(-cE_r)$. We have

$$\mu_* \mathcal{O}_Y(-cE_r) = \mu_* \mathcal{O}_Y(-\bar{c}E_r) \quad \text{with} \quad \bar{c} = \min_{\substack{ap+bq \geq c \\ a,b \geq 0}} (ap+bq).$$

So the unloaded cluster whose associated ideal is $\mu_* \mathcal{O}_Y(-cE_r)$ is $K_{p,q}(\bar{c})$.

Proof of Proposition A.1. We shall argue on the cluster associated to the divisor $-K_{Y|X} + \lfloor \mu^* \xi B \rfloor$. To find the multiplier ideal is equivalent to determine the unloaded corresponding cluster. Let the pull-back of B be $\sum_1^r c_\alpha E_\alpha + B = \boldsymbol{c} \cdot \boldsymbol{E} + B$. Then $-K_{Y|X} + \lfloor \mu^* \xi B \rfloor = \sum_1^r w_\alpha W_\alpha = \boldsymbol{w} \cdot \boldsymbol{w}$, with $\boldsymbol{w} = -\boldsymbol{\omega} + \lfloor \xi \boldsymbol{c} \rfloor \cdot \Pi$ and $\boldsymbol{\omega} = (1, \ldots, 1)$.

Let $P_{\alpha_1}, \ldots, P_{\alpha_l}$ be ordered points that determine a chain of maximal L-shape branches. Then

$$\boldsymbol{w} - \overline{\boldsymbol{w}} = \boldsymbol{w} \cdot {}^t\Pi = \lfloor \xi \boldsymbol{c} \rfloor \cdot \Pi \cdot {}^t\Pi - \boldsymbol{\omega} \cdot {}^t\Pi . \tag{A.2}$$

The matrix $-\Pi \cdot {}^t\Pi$ is the intersection matrix of the strict transforms E_α on the surface Y. So for every $1 \leq j \leq l$,

$$w_{\alpha_j} - \overline{w}_{\alpha_j} = -\lfloor \xi c_{\alpha_{j-1}} \rfloor + (p_{\alpha_j} + 1) \lfloor \xi c_{\alpha_j} \rfloor - \lfloor \xi c_{\alpha_{j+1}} \rfloor + (p_{\alpha_j} - 1)$$

and $\sum_{j=1}^l (w_{\alpha_j} - \overline{w}_{\alpha_j})$ is equal to

$$-\lfloor \xi c_{\alpha_0} \rfloor + p_{\alpha_1} \lfloor \xi c_{\alpha_1} \rfloor + \sum_{j=2}^{l-1} (p_{\alpha_i} - 1) \lfloor \xi c_{\alpha_j} \rfloor + p_{\alpha_l} \lfloor \xi c_{\alpha_l} \rfloor - \lfloor \xi c_{\alpha_{l+1}} \rfloor + \sum_{j=1}^{l} (p_{\alpha_j} - 1).$$

Since $\boldsymbol{c} \cdot \Pi \cdot {}^t\Pi = (0, \ldots, 0, d)$, we have

$$-2 < \sum_{j=1}^{l} (w_{\alpha_j} - \overline{w}_{\alpha_j}) < \sum_{j=1}^{l} p_{\alpha_j} + 2 - l. \tag{A.3}$$

Putting $l = 1$ we observe that if the proximity relation is not satisfied at P_α, then $w_\alpha - \overline{w}_\alpha = -1$. But the unloading procedure of Lemma A.7 at P_α changes the vector $\boldsymbol{w} - \overline{\boldsymbol{w}}$ into the vector $\boldsymbol{w} - \overline{\boldsymbol{w}} + (\Pi \cdot {}^t\Pi)_\alpha$. It follows that the unloading procedure does not change the inequalities in (A.3) for the new cluster. So the associated unloaded cluster satisfies $(*)$. Lemma A.12 gives the result. □

Proof of Corollary A.2. Let $d = \gcd(m,n)$ and $m = dp$ and $n = dq$. Let ξ be a jumping number for B at P and consider the cluster whose associated divisor is the exceptional configuration in $\lfloor \mu^*(\xi B) \rfloor$. The coefficient of E_r, the last strict transform for a log resolution of B at P — the Enriques diagram associated to the configuration of strict transforms above P is $T_{p,q}$ with the weights corresponding to dB_r — must be of the form $ap + bq + 1$, for some $a, b \geq 0$. But the last coefficient is $\lfloor \xi dpq \rfloor - (p + q - 1)$, hence ξ is the minimal rational number such that

$$\lfloor \xi dpq \rfloor = (a+1)p + (b+1)q ,$$

and the result follows. □

EXAMPLES A.13. Let $P \in B$ be a singular point of type A_{2r} locally given by $x^2 + y^{2r+1} = 0$, $r \geq 1$. The Enriques diagram of the minimal log resolution of B at P is $T_{2,2r+1}$ with the weights as shown in Figure 7.

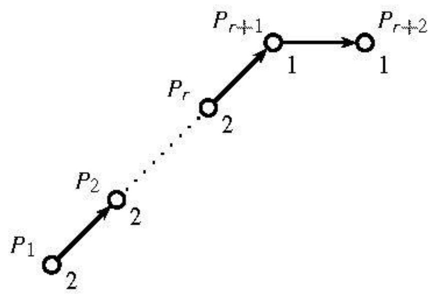

FIGURE 7

By Corollary A.2, the jumping numbers < 1 of B at P are $\xi_a = 1/2 + a/(2r+1)$, with $a = 1, \ldots, r$. Then

$\mathcal{J}(\xi_a \cdot B)_P = \mu_* \mathcal{O}_Y\big(-(2a-1)E_{r+2}\big) = \mu_* \mathcal{O}_Y(-2aE_{r+2}) = \mu_* \mathcal{O}_Y(-W_1 - \cdots - W_a)$ for any a. The corresponding subscheme $Z_{2,2r+1}(\overline{2a-1}) = Z_P^{[a]}$ is the curvilinear subscheme defined by the unloaded cluster $\{P_1, \ldots, P_a\}$.

Let $P \in B$ be a singular point locally given by $x^2 + y^{2r} = 0$, $r \geq 1$. As before the Enriques diagram of the minimal log resolution of B at P is $T_{1,r}$. It consists of r free points with all the weights equal to 2. By Corollary A.2, the jumping numbers < 1 of B at P are $\xi_a = 1/2 + a/(2r)$ with $a = 1, \ldots, r-1$, and by Proposition A.1 the multiplier ideals are $\mathcal{J}(\xi_a \cdot B)_P = \mu_* \mathcal{O}_Y(-aE_r) = \mu_* \mathcal{O}_Y(-W_1 - \cdots - W_a)$. The subscheme $Z_{1,r}(\overline{a})$ is $Z_P^{[a]}$, the curvilinear subscheme corresponding to the unloaded cluster $\{P_1, \ldots, P_a\}$ for any $1 \leq a \leq r-1$.

A.4 OKA'S EXAMPLE AND THE PROOF OF PROPOSITION 4.3

Keeping the set-up and notation of Section 4, suppose that $p < q$. By Theorem 3.1 and Proposition A.1, the irregularity of the pq-multiple plane associated to the curve $C_{p,q}$ is given by

$$q(S) = \sum_{\substack{\alpha,\beta\geq 1\\ \alpha p+\beta q<pq}} h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}_{p,q}(\overline{(\alpha-1)p+(\beta-1)q+1})}(-3+\alpha p+\beta q)\big)\,.$$

The sum consists of $(p-1)(q-1)/2$ terms, and as in the particular case $p = 2$, we shall show that each of them equals 1. For an arbitrary couple (α, β), with $\alpha \geq 2$, we first apply the trace-residual exact sequence $\alpha - 1$ times with respect to C_p. If $\mathcal{Z}$ denotes the subscheme $\mathcal{Z}_{p,q}(\overline{(\alpha-1)p+(\beta-1)q+1})$, using the lemma hereafter, we have the short exact sequence

$$\begin{aligned}0 \longrightarrow \mathcal{I}_{\mathcal{Z}_{p,q}(\overline{(\alpha-2)p+(\beta-1)q+1})}&(-3+\alpha(p-1)+\beta q)\\ &\longrightarrow \mathcal{I}_{\mathcal{Z}}(-3+\alpha p+\beta q) \xrightarrow{\rho} \mathcal{I}_{\mathrm{Tr}_{C_p}\mathcal{Z}}(-3+\alpha p+\beta q) \longrightarrow 0\,.\end{aligned}$$

Let P be any point in the support of $\mathcal{Z}$ and w_1 be the weight of $P = P_1$ in the unloaded cluster $K_{p,q}(\overline{(\alpha-1)p+(\beta-1)q+1})$. The subscheme $\mathcal{Z}$ is contained in $w_1 C_q$, hence using the multiplication with the equation of C_q at the power w_1, the global sections of $\mathcal{O}_{\mathbf{P}^2}(-w_1 q + n)$ live in $H^0(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}}(n))$ for any integer n. Since $\mathrm{Tr}_{C_p}\mathcal{Z} = w_1 C_q\,|_{C_p}$, the global sections of $\mathcal{I}_{\mathrm{Tr}_{C_p}\mathcal{Z}}(-3+n)$ are cut out by the curves of degree $-3+n-w_1 q$, hence $H^0\rho$ is surjective. We conclude that

$$\begin{aligned}\text{(A.4)}\quad h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}_{p,q}(\overline{(\alpha-1)p+(\beta-1)q+1})}&(-3+\alpha p+\beta q)\big)\\ &= h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}_{p,q}(\overline{(\beta-1)q+1})}(-3+p+\beta q)\big)\end{aligned}$$

whenever $\alpha \geq 2$. Then, in case $\beta \geq 2$, we apply $\beta - 1$ times the trace-residual exact sequence with respect to C_q starting with the subscheme $\mathcal{Z} = \mathcal{Z}_{p,q}(\overline{(\beta-1)q+1})$. As before, we have

$$\begin{aligned}0 \to \mathcal{I}_{\mathcal{Z}_{p,q}(\overline{(\beta-2)q+1})}(-3+p+(\beta-1)q) &\to \mathcal{I}_{\mathcal{Z}}(-3+p+\beta q)\\ &\xrightarrow{\rho} \mathcal{I}_{\mathrm{Tr}_{C_q}\mathcal{Z}}(-3+p+\beta q) \to 0\,,\end{aligned}$$

the surjectivity of $H^0\rho$ being given this time by the inclusion $\mathcal{Z} \subset wC_p$, with w the sum of the weights of the points $P_1, P_2, \ldots, P_{a_1}, P_{a_1+1}$ in the cluster $K_{p,q}(\overline{(\beta-1)q+1})$. So

$$\text{(A.5)}\quad h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}_{p,q}(\overline{(\beta-1)q+1})}(-3+p+\beta q)\big) = h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}(\overline{1})}(-3+p+q)\big)\,.$$

Finally, $\mathcal{Z}_{p,q}(\overline{1}) = \bigcup_P P$ and we apply once more the trace-residual exact sequence for this subscheme with respect to C_p to get

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-3+q) \longrightarrow \mathcal{I}_{\mathcal{Z}_{p,q}(\overline{1})}(-3+p+q) \longrightarrow \mathcal{O}_{C_p}(-3+p) \longrightarrow 0\,.$$

Since $q > p$, $h^1\big(\mathbf{P}^2, \mathcal{I}_{\mathcal{Z}_{p,q}(\overline{1})}(-3+p+q)\big) = h^1\big(C_p, \mathcal{O}_{C_p}(-3+p)\big) = 1$. Together with (A.4) and (A.5) this concludes the proof of the proposition.

LEMMA A.14. *Let $Z = Z_{p,q}(\overline{\alpha p + \beta q + 1})$ be the subscheme associated to the unloaded cluster $K_{p,q}(\overline{\alpha p + \beta q + 1})$ and centered at a point of intersection of C_p and C_q. If $\alpha \geq 1$, then* $\mathrm{Res}_{C_p} Z = Z_{p,q}(\overline{(\alpha-1)p + \beta q + 1})$, *and if $\beta \geq 1$, then* $\mathrm{Res}_{C_q} Z = Z_{p,q}(\overline{\alpha p + (\beta - 1)q + 1})$.

Proof. We first show that if $K_{p,q}(\overline{\alpha p + \beta q + 1}) = \{P_1^{w_1}, \dots, P_r^{w_r}\}$, then the subscheme $\mathrm{Res}_C Z$ corresponds to the unloaded cluster associated to $K = \{P_1^{w_1 - 1}, \dots, P_r^{w_r}\}$. To see this, let us denote by ε the blowing up of the plane at $P = P_1$ and by μ' the sequence of the remaining blowing ups that compose $\mu : X \xrightarrow{\mu'} X_2 = \mathrm{Bl}_P \mathbf{P}^2 \xrightarrow{\varepsilon} X_1 = \mathbf{P}^2$. Then

$$\mathcal{I}_Z = \mu_* \mathcal{O}_X(-D_K) = \varepsilon_*\big(\mathcal{O}_{\mathrm{Bl}_P \mathbf{P}^2}(-w_1 W_1^{(2)}) \otimes \mu'_* \mathcal{O}_X(-D_K + w_1 W_1)\big)\,.$$

The ideal $\mu'_* \mathcal{O}_X(-D_K + w_1 W_1)$ is associated to the cluster $K' = \{P_2^{w_2}, \dots, P_r^{w_r}\}$ centered at P_2. If ε is given locally around P_2 by $x = x'y'$ and $y = y'$, then the equation of the exceptional divisor $E_1^{(2)} = W_1^{(2)} \subset \mathrm{Bl}_P \mathbf{P}^2$ is $y' = 0$. It follows that

$$\text{(A.6)} \qquad \begin{aligned}(\mathcal{I}_Z : \mathcal{I}_{C_p}) &= \big(\varepsilon_*(\mathcal{O}_{\mathrm{Bl}_P \mathbf{P}^2}(-w_1 W_1^{(2)}) \otimes \mu'_* \mathcal{O}_X(-D_K + w_1 W_1)) : \mathcal{I}_{W_1^{(2)}}\big) \\ &= \varepsilon_*\big(\mathcal{O}_{\mathrm{Bl}_P \mathbf{P}^2}(-(w_1 - 1) W_1^{(2)}) \otimes \mu'_* \mathcal{O}_X(-D_K + w_1 W_1)\big)\,,\end{aligned}$$

hence the result. Next, suppose that $\overline{\alpha p + \beta q + 1} = ap + bq$. From the proof of Lemma A.12, since the cluster $\{P_1^{w_1}, \dots, P_r^{w_r}\}$ satisfies condition $(*)$, it follows that K satisfies this condition too and hence $\tilde{K}$ is of the type $K_{p,q}(\bar{c})$ with

$$c = \varphi\big((w_1 - 1)W_1 + \sum_{\alpha \geq 2} W_\alpha\big) = \varphi\big(\sum_\alpha W_\alpha\big) - \varphi(W_1) = ap + bq - p = (a-1)p + bq\,.$$

Here φ is as before the projection $\varphi\colon \bigoplus_\alpha \mathbf{Z}E_\alpha \to \mathbf{Z} \simeq \mathbf{Z}E_r$. So $\bar{c} = c$ and $\mathrm{Res}_{C_p} Z = Z_{p,q}((a-1)p + bq)$. To finish the proof of the first assertion, it is sufficient to show that $\overline{(\alpha - 1)p + \beta q + 1} = (a-1)p + bq$, where $\alpha \geq 1$ and $\overline{\alpha p + \beta q + 1} = ap + bq$. But this is clear, since if there existed non-negative integers a', b' such that

$$(a-1)p+bq > a'p+b'q \geq (\alpha-1)p+\beta q+1\,,$$

then $\overline{\alpha p+\beta q+1}$ would be equal to $(a'+1)p+b'q$.

The proof of the second assertion is similar; the argument in formula (A.6) has to be repeated a_1+1 times, i.e. for all the free points of the Enriques diagram. □

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