

# The Friedlander-Milnor conjecture

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### THE FRIEDLANDER–MILNOR CONJECTURE

by Eric M. FRIEDLANDER

The conjecture of the title of this note has resisted 40 years of effort and remains not only unsolved but also lacking in a plausible means of either proof or counter-example.

The original form of this conjecture is one I struggled with during my days at Princeton in the early 1970's:

CONJECTURE 32.1. *Let  $G(\mathbf{C})$  be a complex reductive algebraic group and let  $G(\mathbf{C})^\delta$  denote this group viewed as a discrete group. Then the map on classifying spaces of the continuous (identity) group homomorphism*

$$i: G(\mathbf{C})^\delta \rightarrow G(\mathbf{C})$$

*induces an isomorphism in cohomology with finite coefficients  $\mathbf{Z}/n$  for any  $n \geq 0$ :*

$$i^*: H^*(BG(\mathbf{C}), \mathbf{Z}/n) \xrightarrow{\simeq} H^*(G(\mathbf{C})^\delta, \mathbf{Z}/n).$$

Conjecture 32.1 is easily seen to be true for a torus (i.e.,  $G = \mathbf{G}_m^{\times r}$  for some  $r > 0$ ), but even the simplest non-trivial case (that of  $G = \mathrm{SL}_2$ ) remains inaccessible.

Guido and I published 5 papers together, all in some sense connected with this conjecture. We used the integral form  $G_{\mathbf{Z}}$  of a complex reductive algebraic group (which is a group scheme over  $\mathrm{Spec} \mathbf{Z}$ ) in order to form the group  $G(F)$  of points of  $G$  with values in a field  $F$ . Most of our joint work investigated various relations between  $G(\mathbf{C})$  and  $G(F)$ , the case  $F = \overline{\mathbf{F}}_p$  (the algebraic closure of a prime field  $\mathbf{F}_p$ ) being of special interest.

One knows from considerations of étale cohomology that the cohomology of  $BG(\mathbf{C})$  with  $\mathbf{Z}/n$  coefficients is naturally isomorphic to that of the étale

homotopy classifying space of the algebraic group  $G_F$  for  $F$  algebraically closed of characteristic  $p \geq 0$ :

$$H^*(BG(\mathbf{C}), \mathbf{Z}/n) \simeq H^*((BG_F)_{\text{et}}, \mathbf{Z}/n), \quad \text{provided that } (p, n) = 1.$$

This enables one to construct a map  $H^*(BG(\mathbf{C}), \mathbf{Z}/n) \rightarrow H^*(G(F), \mathbf{Z}/n)$  relating the cohomology with mod- $n$  coefficients of the classifying space of  $G(\mathbf{C})$  with the cohomology with mod- $n$  coefficients of the discrete group  $G(F)$  for any field  $F$ .

The following is a generalization of Conjecture 32.1, one that appears likely to be true if and only if Conjecture 32.1 is valid.

**CONJECTURE 32.2.** *Let  $G(\mathbf{C})$  be a complex reductive algebraic group, let  $n > 0$  be a positive integer, and let  $p$  denote either 0 or a prime which does not divide  $n$ . Then for any algebraically closed field  $F$  of characteristic  $p$ , the comparison of the cohomology of  $BG(\mathbf{C})$  and  $G(F)$  determines an isomorphism*

$$H^*(G(F), \mathbf{Z}/n) \simeq H^*(BG(\mathbf{C}), \mathbf{Z}/n).$$

In our first paper together [1], Guido and I began our investigation of “locally finite approximations” of Lie groups. We also formulated the following conjecture and proved it equivalent to Conjecture 32.2.

**CONJECTURE 32.3.** *Let  $F$  be an algebraically closed field of characteristic  $p \geq 0$  and let  $n > 0$  be a positive integer not divisible by  $p$  if  $p > 0$ . Then Conjecture 32.2 is valid for  $G(F)$  if and only for every  $0 \neq x \in H^*(G(F), \mathbf{Z}/n)$ , there exists some finite subgroup  $\pi \subset G(F)$  such that  $x$  restricts non-trivially to  $H^*(\pi, \mathbf{Z}/n)$ .*

The most familiar form of the “Friedlander–Milnor Conjecture” is that formulated by John Milnor in [2]. In that paper, Milnor verifies this conjecture for solvable groups.

**CONJECTURE 32.4.** *Let  $G$  be a Lie group with finitely many components and let  $G^\delta$  denote the same group now viewed as a discrete group. Then for any integer  $n > 0$ , the continuous (identity) map  $i: G^\delta \rightarrow G$  induces an isomorphism on cohomology with mod- $n$  coefficients:*

$$i^*: H^*(BG, \mathbf{Z}/n) \xrightarrow{i^*} H^*(G^\delta, \mathbf{Z}/n).$$

We remark that the most substantial progress to date on these conjectures is due to Andrei Suslin, who proves a “stable” version of Conjectures 32.1 and 32.2 in [3].

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